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The Second Order Upper Bound for the Ground Energy of a Bose Gas

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Abstract Consider N bosons in a finite box $\Lambda = [0, L]^3 \subset \mathbf{R}^3$ interacting via a two-body smooth repulsive short range potential. We construct a variational state which gives the following upper bound on the ground state energy per particle

$$\overline{\lim}_{\rho\to 0}\overline{\lim}_{L\to\infty,\, N/L^3\to\rho}\left(\frac{e_0(\rho)-4\pi a\rho}{(4\pi a)^{5/2}(\rho)^{3/2}}\right)\le \frac{16}{15\pi^2},$$

where a is the scattering length of the potential. Previously, an upper bound of the form $C16/15\pi^2$ for some constant $C > 1$ was obtained in (Erdös et al. in Phys. Rev. A 78:053627, 2008). Our result proves the upper bound of the prediction by Lee and Yang (Phys. Rev. 105(3):1119–1120, 1957) and Lee et al. (Phys. Rev. 106(6):1135–1145, 1957).

Keywords Bose gas · Bogoliubov transformation · Variational principle

AMS Subject Classification 82B10

1 Introduction

The ground state energy is a fundamental property of a quantum system and it has been intensively studied since the invention of the quantum mechanics. The recent progresses in experiments for the Bose-Einstein condensation have inspired re-examination of the theoretic foundation concerning the Bose system and, in particular, its ground state energy. In the low density limit, the leading term of the ground state energy per particle was identified rigorously by Dyson (upper bound) [3] and Lieb-Yngvason (lower bound) [14] to be $4\pi a\varrho$, where a is the scattering length of the two-body potential and ϱ is the density. The famous second order correction to this leading term was first computed by Lee-Yang [10] (see also

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Lee-Huang-Yang [9] and the recent paper by Yang [16] for results in other dimensions). To describe this prediction, we now fix our notations: Consider N interacting bosons in a finite box $\Lambda = [0, L]^3 \subset \mathbf{R}^3$ with periodic boundary conditions. The two-body interaction is given by a smooth nonnegative potential V of fast decay. The Lee-Yang's prediction of the energy per particle up to the second order is given by

$$e_0(\rho) = 4\pi \varrho a \left[1 + \frac{128}{15\sqrt{\pi}} (\varrho a^3)^{1/2} + \cdots \right]. \tag{1.1}$$

The approach by Lee-Yang [10] is based on the pseudo-potential approximation [7, 9] and the "binary collision expansion method" [9]. One can also obtain (1.1) by performing the Bogoliubov [1, 2] approximation and then replacing the integral of the potential by its scattering length [8]. Another derivation of (1.1) was later given by Lieb [11] using a self-consistent closure assumption for the hierarchy of correlation functions.

In the recent paper [4], the potential V was replaced by λV_0 for some fixed function V_0 and λ is small. A variational state was constructed to yield the rigorous upper bound

$$e_0(\rho) \le 4\pi \varrho a \left[1 + \frac{128}{15\sqrt{\pi}} (\varrho a^3)^{1/2} S_\lambda \right] + O(\varrho^2 |\log \varrho|) \tag{1.2}$$

with $S_\lambda \le 1 + C\lambda$. In the limit $\lambda \to 0$, one recovers the prediction of Lee-Yang [10] and Lee-Huang-Yang [9]. The trial state in [4] does not have a fixed number of particles, and is a state in the Fock space with expected number of particles N (presumably a trial state with a fixed number of particles can be constructed with a similar idea). The trial state in [4] is similar to the trial state used by Girardeau and Arnowitt [5] and recently by Solovej [15]; it is of the form

$$\exp\left[|\Lambda|^{-1} \sum_k c_k a_k^\dagger a_{-k}^\dagger a_0 \, a_0 \; + \sqrt{N_0} a_0^\dagger \right] |0\rangle \tag{1.3}$$

where c_k and N_0 have to be chosen carefully to give the correct asymptotic in energy. This state captures the idea that particle pairs of opposite momenta are created from the sea of condensate consisting of zero momentum particles. It is believed that this type of trial state gives the ground state energy consistent with the Bogoliubov approximation. In the case of Bose gas, the Bogoliubov approximation yields the correct energy up to the order $\rho^{3/2}$, but the constant is correct only in the semiclassical limit—consistent with the calculation using the trial state (1.3). It should be noted that the Bogoliubov approximation gives the correct "correlation energy" in several setting including the one and two component charged Boson gases [12, 13, 15] and the Bose gas in large density-weak potential limit [6].

For the Bose gas in low density, the result of [4] suggests to correct the error by renormalizing the propagator. Unfortunately, it is difficult to implement this idea. Our main observation is to relax the concept of condensates by allowing particle pairs to have nonzero total momenta. More precisely, we consider a trial state of the form

$$\begin{aligned} \exp\Bigg[& |\Lambda|^{-1} \sum_k \sum_{v \sim \sqrt{\rho}} 2\sqrt{\lambda_{k+v/2}\lambda_{-k+v/2}}\, a_{k+v/2}^\dagger a_{-k+v/2}^\dagger a_v \, a_0 \\ & + |\Lambda|^{-1} \sum_k c_k a_k^\dagger a_{-k}^\dagger a_0 \, a_0 \; + \sqrt{N_0} a_0^\dagger \Bigg] |0\rangle \end{aligned} \tag{1.4}$$

for suitably chosen c and λ. Notice that the total momentum of the pair, v, is required to be of order $\rho^{1/2}$ and the constant 2 comes from the ordering of $a_v a_0$. We shall make further simplification that $\lambda_k = c_k$. Even with this simplification, however, this state is still too complicated. We will extract some properties from this representation and define an N particle trial state whose energy is given by the Lee-Yang's prediction up to the second order term. Details will be given in Sect. 3. Our result shows that, in order to obtain the second order energy, the typical ansatz for the Bogoliubov approximation should be extended to allow pair particles with nonzero momenta. This also suggests that the Bogoliubov approximation has to be modified in order to yield the correct energy of the low density Bose gas to the second order.

2 Notations and Main Results

Let $\Lambda = [0, L]^3 \subset \mathbb{R}^3$ be a cube with periodic boundary conditions with the dual space $\Lambda^* := (\frac{2\pi}{L}\mathbb{Z})^3$. The Fourier transform is defined as

$$W_p := \hat{W}(p) = \int_{x\in\mathbb{R}^3} e^{-ipx} W(x)\,\mathrm{d}x, \qquad W(x) = \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*} e^{ipx} W_p.$$

Here we have used the convention to denote the Fourier transform of a function W at the momentum p by W_p instead of $\widehat{W}(p)$ to avoid too heavy notations. Since the summation of p is always restricted to Λ^*, we will not explicitly specify it.

We will use the bosonic operators with the commutator relations

$$[a_p, a_q^\dagger] = a_p a_q^\dagger - a_q^\dagger a_p = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

The two body interaction is given by a smooth, symmetric non-negative function $V(x)$ of fast decay. Clearly, in the Fourier space, we have $V_u = V_{-u} = \bar{V}_u$. Furthermore, we assume that the potential V is small so that the Born series converges. The Hamiltonian of the many-body systems with the potential V and the periodic boundary condition is thus given by

$$H = \sum_p p^2 a_p^\dagger a_p + \frac{1}{|\Lambda|}\sum_{p,q,u} V_u a_p^\dagger a_q^\dagger a_{p-u} a_{q+u}. \tag{2.1}$$

Let $1 - w$ be the zero energy scattering solution

$$-\Delta(1 - w) + V(1 - w) = 0$$

with $0 \le w < 1$ and $w(x) \to 0$ as $|x| \to \infty$. Then the scattering length is given by the formula

$$a := \frac{1}{4\pi}\int_{\mathbb{R}^3} V(x)(1 - w(x))\,\mathrm{d}x.$$

Introduce g_0, whose meaning will be explained later on, to denote the quantity

$$g_0 = 4\pi a.$$

Let $\mathcal{H}_N$ be the Hilbert space of N bosons. Denote by $\rho_N = N/\Lambda$ the density of the system. The ground state energy of the Hamiltonian (2.1) in $\mathcal{H}_N$ is given by

$$E_0^P(\rho, \Lambda) = \inf spec\, H_{\mathcal{H}_N}$$

and the ground state energy per particle is $e_0^P(\rho,\Lambda)=E_0^P(\rho,\Lambda)/N$. We can also consider other boundary conditions, e.g., $e_0^D(\rho,\Lambda)$ is the Dirichlet boundary condition ground state energy per particle.

In this paper, we will always take the limit $L\to\infty$ so that the density $\rho_N\to\rho$ for some fixed density ρ. From now on, we will use $\lim_{L\to\infty}$ for the more complicated notation $\lim_{L\to\infty,\,N/L^3\to\rho}$. We now state the main result of this paper.

Theorem 2.1 *Suppose the potential V is smooth, symmetric, nonnegative with fast decay and sufficiently small so that the Born series converges. Then the ground state energy per particle satisfies the upper bound*

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{L\to\infty}}\left(\frac{e_0^P(\rho,\Lambda)-g_0\rho}{g_0^{5/2}\rho^{3/2}}\right)\le\frac{16}{15\pi^2}. \tag{2.2}$$

Although we state the theorem in the form of limit $\rho\to 0$, an error bound is available from the proof. We avoid stating such an estimate to simplify the notations and proofs. Our result holds also for Dirichlet boundary condition.

2.1 Reduction to Small Torus with Periodic Boundary Conditions

To prove Theorem 2.1, we only need to construct a trial state $\Psi(\rho,\Lambda)$ satisfying the boundary condition and

$$\overline{\lim_{\rho\to 0}}\,\lim_{\Lambda\to\infty}\left(\frac{\langle H_N\rangle_\Psi N^{-1}-g_0\rho}{g_0^{5/2}\rho^{3/2}}\right)\le\frac{16}{15\pi^2}. \tag{2.3}$$

The first step is to construct a trial state with a Dirichlet boundary condition in a cube of order slightly bigger than ρ^{-1}.

Lemma 2.1 *For density ρ small enough, there exist $L\sim\rho^{-25/24}$ and a trial state Ψ of N ($N=\rho L^3$) particles on $\Lambda=[0,L]^3$ satisfying the Dirichlet boundary condition and*

$$\overline{\lim_{\rho\to 0}}\left(\frac{\langle H_N\rangle_\Psi N^{-1}-g_0\rho}{g_0^{5/2}\rho^{3/2}}\right)\le\frac{16}{15\pi^2}. \tag{2.4}$$

Once we have a trial state with the Dirichlet boundary condition, we can duplicate it so that a trial state can be constructed for cubes with linear dimension $\gg\rho^{-25/24}$. This proves Theorem 2.1.

The next lemma shows that a Dirichlet boundary condition trial state with correct energy can be obtained from a periodic one.

Lemma 2.2 *Recall the ground state energies per particle $e_0^D(\rho,\Lambda)$ and $e_0^P(\rho,\Lambda)$ for the Dirichlet and periodic boundary condition. Let $\Lambda=[0,L]^3$ and $L=\rho^{-25/24}$. Suppose the energy for the periodic boundary condition satisfies that*

$$\overline{\lim_{\rho\to 0}}\left(\frac{e_0^P(\rho,\Lambda)-g_0\rho}{g_0^{5/2}\rho^{3/2}}\right)\le\frac{16}{15\pi^2}. \tag{2.5}$$

Then for $\widetilde{\Lambda} = [0, \widetilde{L}]^3$, $\widetilde{L} = L(1 + 2\rho^{25/48})$ *and* $\widetilde{\rho} = \rho L^3/\widetilde{L}^3$, *the following estimate for the energy of the Dirichlet boundary condition holds*:

$$\overline{\lim_{\rho\to 0}}\left(\frac{e_0^D(\widetilde{\rho}, \widetilde{\Lambda}) - g_0\widetilde{\rho}}{g_0^{5/2}\widetilde{\rho}^{3/2}}\right) \le \frac{16}{15\pi^2}. \tag{2.6}$$

The construction of a periodic trial state yielding the correct energy upper bound is the core of this paper. We state it as the following theorem.

Theorem 2.2 *There exists a periodic trial state* Ψ *of* N *particles on* $\Lambda = [0, L]^3$, $L = \rho^{-25/24}$ *such that* $(N = |\Lambda|\rho)$

$$\overline{\lim_{\rho\to 0}}\left(\frac{\langle H_N\rangle_\Psi N^{-1} - g_0\rho}{g_0^{5/2}\rho^{3/2}}\right) \le \frac{16}{15\pi^2}. \tag{2.7}$$

This paper is organized as follows: In Sect. 3, we define rigorously the trial state. In Sect. 4, we outline the lemmas needed to prove Theorem 2.2. In Sect. 5, we estimate the number of particles in the condensate and various momentum regimes. These estimates are the building blocks for all other estimates later on. In Sect. 6, we estimate the kinetic energy. The potential energy is estimated in Sects. 7–11. Finally in Sect. 12, we prove the reduction to the periodic boundary condition, i.e., Lemma 2.2. This proof follows a standard approach and only a sketch will be given.

3 Definition of the Trial State

We now give a formal definition of the trial state. This somehow abstract definition will be explained later on. We first identify four regions in the momentum space Λ^* which are relevant to the construction of the trial state: P_0 for the condensate, P_L for the low momenta, which are of the order $\rho^{1/2}$; P_H for momenta of order one, and P_I the region between P_L and P_H.

Definition 3.1 *Define four subsets of momentum space*: P_0, P_L, P_I *and* P_H.

$$\begin{aligned} P_0 &\equiv \{p = 0\}, \\ P_L &\equiv \left\{p \in \Lambda^* | \varepsilon_L \rho^{1/2} \le |p| \le \eta_L^{-1}\rho^{1/2}\right\}, \\ P_I &\equiv \left\{p \in \Lambda^* | \eta_L^{-1}\rho^{1/2} < |p| \le \varepsilon_H\right\}, \\ P_H &\equiv \left\{p \in \Lambda^* | \varepsilon_H < |p|\right\}, \end{aligned} \tag{3.1}$$

where the parameters are chosen so that

$$\varepsilon_L, \eta_L, \varepsilon_H \equiv \rho^\eta \quad \text{and} \quad \eta \equiv 1/200. \tag{3.2}$$

Denote by $P = P_0 \cup P_L \cup P_I \cup P_H$.

We remark that the momenta between P_0 and P_L are irrelevant to our construction. Next, we need a notation for the collection of states with N particles.

Definition 3.2 *Let $\widetilde{M}$ be the set of all functions $\alpha : P \to \mathbb{N} \cup 0$ such that*

$$\sum_{k \in P} \alpha(k) = N. \tag{3.3}$$

For any $\alpha \in \widetilde{M}$, denote by $|\alpha\rangle \in \mathcal{H}_N$ the unique state (in this case, an N-particle wave function) defined by the map α

$$|\alpha\rangle = C \prod_{k \in P} (a_k^\dagger)^{\alpha(k)} |0\rangle,$$

where the positive constant C is chosen so that $|\alpha\rangle$ is L_2 normalized. Define α_{free} as $\alpha_{\text{free}}(k) = N\delta_{0,k}$.

Clearly, we have

$$a_k^\dagger a_k |\alpha\rangle = \alpha(k)|\alpha\rangle, \quad \forall k \in P. \tag{3.4}$$

Definition 3.3 *We define two relations between functions in $\widetilde{M}$:*

1. *Strict pair creation of momentum k: Denote by $\beta := \mathcal{A}^k \alpha$ if β is generated by creating a pair of particles with momenta k and $-k$, i.e.,*

$$\beta(p) = \begin{cases} \alpha(p) - 2, & p = 0, \\ \alpha(p) + 1, & p = \pm k, \\ \alpha(p), & \textit{others}. \end{cases} \tag{3.5}$$

   In terms of states, we have

$$|\beta\rangle = C a_k^+ a_{-k}^+ a_0^2 |\alpha\rangle$$

   where C is a positive constant so that the state $|\beta\rangle$ is normalized.

2. *Soft pair creation with total momentum u and difference $2k$: Denote by $\beta = \mathcal{A}^{u,k}\alpha$ if β is generated by creating two particles with high momenta $\pm k + u/2 \in P_H$ so that the total momentum u is in P_L, i.e.,*

$$\beta(p) = \begin{cases} \alpha(p) - 1, & p = 0 \text{ or } u, \\ \alpha(p) + 1, & p = \pm k + u/2, \\ \alpha(p), & \textit{others}. \end{cases} \tag{3.6}$$

   Notice that $\mathcal{A}^{u,k}\alpha$ is defined only if $\pm k + u/2 \in P_H$. In terms of states, we have

$$|\beta\rangle = C a_{k+u/2}^+ a_{-k+u/2}^+ a_0 a_u |\alpha\rangle$$

   where C is the normalization constant. Since $\beta(p)$ has to be nonnegative, the state $\mathcal{A}^k\alpha$ or $\mathcal{A}^{u,k}\alpha$ is not defined for all α or k, u.

Define D_α to be the set all possible derivations of α from the previous two operations:

$$D_\alpha = \left\{\mathcal{A}^{u,k}\alpha \in \widetilde{M}\right\} \cup \left\{\mathcal{A}^k\alpha \in \widetilde{M}\right\}. \tag{3.7}$$

Our trial state will be of the form $\sum_{\alpha \in \widetilde{M}} f(\alpha)|\alpha\rangle$ where f is supported in a subset of $\widetilde{M}$ which we now define.

Definition 3.4 *Fix a large real number k_c. We define M as the smallest subset of $\widetilde{M}$ such that*

1. $\alpha_{\text{free}} \in M$.
2. *M is closed under strict pair creation provided the momentum $u \in P_I \cup P_H$, i.e., if $\alpha \in M$ and $\mathcal{A}^u\alpha \in \widetilde{M}$ then $\mathcal{A}^u\alpha \in M$.*
3. *M is closed under strict pair creation provided the momentum $u \in P_L$ and $\max\{\alpha(u), \alpha(-u)\} < m_c$, i.e., if $\alpha \in M$ and $\mathcal{A}^u\alpha \in \widetilde{M}$, then $\mathcal{A}^u\alpha \in M$. Here we choose m_c as*

$$m_c \equiv \rho^{-\eta} = \rho^{-1/200}. \tag{3.8}$$

4. *M is closed under soft pair creation from states with perfect pairing of momenta u and $-u$. More precisely, for $u \in P_L$ with $\alpha(u) = \alpha(-u)$, if $\alpha \in M$, $\mathcal{A}^{u,k}\alpha \in \widetilde{M}$ and*

$$\varepsilon_H \le |\pm k + u/2| \le k_c,$$

then $\mathcal{A}^{u,k}\alpha \in M$.

The set M is unique since the intersection of two such sets M_1 and M_2 satisfies all four conditions.

For any $u \in P_L$, we define the set of states with symmetric (asymmetric resp.) pair particles of momenta $u, -u$ by M_u^s (M_u^a resp.):

$$M_u^s \equiv \{\alpha \in M | \alpha(u) = \alpha(-u)\},$$
$$M_u^a \equiv \{\alpha \in M | \alpha(u) \neq \alpha(-u)\}. \tag{3.9}$$

Denote by $\alpha^*(u)$ the maximum of $\alpha(u)$ and $\alpha(-u)$:

$$\alpha^*(u) = \max\{\alpha(u), \alpha(-u)\}. \tag{3.10}$$

Since soft pair creation was allowed only from momenta in P_L and the final momenta are in P_H, we have

$$\alpha^*(u) - \alpha(u) \in \{0, 1\}, \quad \alpha(-u) = \alpha(u), \quad \text{for all } u \in P_I.$$

Before defining the weight $f(\alpha)$, we introduce several quantities related to the scattering equation. In the momentum space, the scattering equation is given by ($p \in \mathbb{R}^3$)

$$-p^2 w_p + V_p - \int_r V_{p-r} w_r = 0, \quad \forall p \neq 0. \tag{3.11}$$

Let g be the function

$$g(x) := V(x)(1 - w(x)). \tag{3.12}$$

Then the scattering equation in momentum space takes the form

$$g_p = p^2 w_p \quad \forall p \neq 0. \tag{3.13}$$

One can check $4\pi a = g_0$ this explains the notation g_0 used in Theorem 2.1 and Theorem 2.2.

Definition 3.5 *Define for all* $\varepsilon \neq 0$

$$\rho_\varepsilon \equiv \rho_0 + \varepsilon \rho^{3/2}, \quad \rho_0 := \rho - \frac{1}{3\pi^2}(g_0)^{3/2}\rho^{3/2}, \tag{3.14}$$

where ρ_0 *will be the approximate density of the condensate. Define the "chemical potential"* λ *by*

$$\lambda_k = \begin{cases} \frac{1-\sqrt{1+4\rho g_0|k|^{-2}}}{1+\sqrt{1+4\rho g_0|k|^{-2}}}\rho^{-1}, & k \in P_L, \\ -w_k, & k \in P_I \cup P_H. \end{cases} \tag{3.15}$$

One can check that, to the leading order, λ is given by

$$\rho\lambda_k \equiv \frac{1-\sqrt{1+4\rho g_k|k|^{-2}}}{1+\sqrt{1+4\rho g_k|k|^{-2}}}. \tag{3.16}$$

Notice that λ_k is real number and can be negative.

Definition 3.6 (The Trial State) *Let* Ψ *be defined by*

$$\Psi \equiv \sum_{\alpha \in M} f(\alpha)|\alpha\rangle \tag{3.17}$$

where the coefficient f *is given by*

$$f(\alpha) = C_N \sqrt{\frac{|\Lambda|^{\alpha(0)}}{\alpha(0)!}} \prod_{k\neq 0}(\sqrt{\lambda_k})^{\alpha(k)} \prod_{u\in P_L, \alpha^*(u)-\alpha(u)=1} \sqrt{\frac{4\alpha^*(u)\lambda_u}{|\Lambda|}}. \tag{3.18}$$

Here we follow the convention $\sqrt{x} = \sqrt{|x|}i$ *for* $x < 0$. *For convenience, we define* $f(\alpha) = 0$ *for* $\alpha \notin M$. *The constant* C_N *is chosen so that* Ψ *is* L_2 *normalized, i.e.,*

$$\langle \Psi|\Psi\rangle = 1.$$

Theorem 3.1 *Suppose* $\Lambda = [0, L]^3$ *and* $L = \rho^{-25/24}$. *Then the trial state* Ψ *in* (3.18) *satisfies the estimate*

$$\overline{\lim_{k_c\to\infty}}\ \overline{\lim_{\rho\to 0}} \left(\frac{\langle H_N\rangle_\Psi N^{-1} - g_0\rho}{g_0^{5/2}\rho^{3/2}}\right) \le \frac{16}{15\pi^2}, \tag{3.19}$$

where k_c *is given in Definition* 3.4. *We recall that* $m_c^{-1}, \varepsilon_L, \eta_L, \varepsilon_H$ *are chosen as a small power of* ρ *in* (3.2) *and* (3.8).

3.1 Heuristic Derivation of the Trial State

We now give a heuristic idea for the construction of the trial state. Fix an ordering of momenta in Λ^* so that the first one is the zero momentum. We will use the occupation number representation so that

$$|n_1,\ n_2, \ldots\rangle \tag{3.20}$$

represents the normalized state with n_i particles of momentum k_i. For example,

$$|N, 0, 0, \ldots\rangle = \frac{1}{\sqrt{N!}}(a_0^\dagger)^N |0\rangle.$$

Recall that we would like to generate a state of the form in (1.4). A slightly modified one is

$$\exp\Bigg[|\Lambda|^{-1}\sum_k \sum_{v\sim\sqrt{\rho}} 2\sqrt{\lambda_{k+v/2}\lambda_{-k+v/2}}\,a^\dagger_{k+v/2}a^\dagger_{-k+v/2}a_v\, a_0 \\ + |\Lambda|^{-1}\sum_k \lambda_k a_k^\dagger a_{-k}^\dagger a_0\, a_0\Bigg]|N, 0, 0, \ldots\rangle. \tag{3.21}$$

We now expand the exponential and require that $a^\dagger_{k+v/2}a^\dagger_{-k+v/2}a_v\, a_0$ to appear at most once. The rationale of this assumption is that the soft pair creation is a rare event and thus we can neglect higher order terms. Our trial state is thus a sum of the following state parametrized by $k_1, \ldots, k_s$, $n_1, \ldots, n_s$, $k_1', \ldots, k_t'$ and $v_1, \ldots, v_t$:

$$\text{const.} \prod_{j=1}^t \sqrt{4\lambda_{k_j'+v_j/2}\lambda_{-k_j'+v_j/2}} \prod_{i=1}^s (\lambda_{k_i})^{n_i} |\alpha\rangle \tag{3.22}$$

where

$$|\alpha\rangle = \text{const.}\, |\Lambda|^{-t-\sum_{i=1}^s n_i} \prod_{j=1}^t a^\dagger_{\frac{v_j}{2}+k_j'} a^\dagger_{\frac{v_j}{2}-k_j'} a_{v_j} a_0 \\ \times \prod_{i=1}^s \frac{1}{n_i!}\left(a^\dagger_{k_i} a^\dagger_{-k_i} a_0\, a_0\right)^{n_i} |N, 0, \ldots\rangle. \tag{3.23}$$

Here we have chosen the constant so that the norm of $|\alpha\rangle$ is one. We also require that $v_i + v_j \neq 0$ for $1 \leq i, j \leq t$ since $v_i + v_j = 0$ is a higher order event.

We further make the simplifying assumption that $v_i \in P_L$. Observe now that the state $|\alpha\rangle$ can be obtained from strict and soft pair creations. This explains the core idea behind the definition of M in Definition 3.4. Other restrictions in the definition were mostly due to various cutoffs needed in the estimates. Finally, up to factors depending only on Λ and N, the coefficient in (3.22) gives $f(\alpha)$ in (3.18). Notice all factors depending on s, t, n_i were already included in $|\alpha\rangle$.

The choice of λ is much more complicated. To the first approximation, λ can be obtain from the work of [4]. We thus use this choice to identify the error terms. Once this is done, we optimize the main terms and this leads to the current definition of λ. Notice that, since our trial state is different, there are more main terms than in [4].

4 Proof of Theorem 2.2

Proof Our goal is to prove

$$\overline{\lim_{k_c\to\infty}}\left(\overline{\lim_{\rho\to 0}}\left(\frac{|\Lambda|^{-1}\langle H\rangle_\Psi - g_0\rho^2}{\rho^{5/2}}\right)\right) \leq \frac{16}{15\pi^2} g_0^{5/2}. \tag{4.1}$$

Here $g_0=4\pi a$, $\langle H\rangle_\Psi=\langle\Psi|H|\Psi\rangle$. We decompose the Hamiltonian as follows:

$$H=\sum_{i=1}^{N}-\Delta_i+H_{S1}+H_{S2}+H_{S3}+H_{A1}+H_{A2}, \tag{4.2}$$

where

1. H_{S1} is the part of interaction that annihilates two particles and creates the same two particles, i.e.,
$$H_{S1}=|\Lambda|^{-1}\sum_u V_0 a_u^\dagger a_u^\dagger a_u a_u+|\Lambda|^{-1}\sum_{u\neq v}(V_{u-v}+V_0)a_u^\dagger a_v^\dagger a_u a_v. \tag{4.3}$$
2. H_{S2} is the interaction between the condensate and strict pairs, i.e.,
$$H_{S2}=|\Lambda|^{-1}\sum_{u\neq 0}V_u a_u^\dagger a_{-u}^\dagger a_0 a_0+C.C. \tag{4.4}$$
3. H_{S3} is the part of interaction that strict pairs are involved, i.e.,
$$H_{S3}=|\Lambda|^{-1}\sum_{u,v\neq 0,u\neq v}V_{u-v}a_u^\dagger a_{-u}^\dagger a_v a_{-v}. \tag{4.5}$$
4. H_{A1} is the part of the interaction that one and only one condensate particle is involved i.e.,
$$H_{A1}=|\Lambda|^{-1}\sum_{v_1,v_2,v_3\neq 0}2V_{v_2}a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}+C.C. \tag{4.6}$$
5. H_{A2} is the part of the interaction which is not counted in H_{S1} and there is no condensate nor strict pair involved i.e.,
$$H_{A2}=|\Lambda|^{-1}\sum_{v_i\neq 0,v_1+v_2\neq 0,\{v_1,v_2\}\neq\{v_3,v_4\}}V_{v_1-v_3}a_{v_1}^\dagger a_{v_2}^\dagger a_{v_3}a_{v_4}. \tag{4.7}$$

The estimates for the energies of these components are stated as the following lemmas, which will be proved in later sections.

Lemma 4.1 *The total kinetic energy is bounded above by*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\left\langle\sum_{i=1}^{N}-\Delta_i\right\rangle_\Psi-\rho_0^2\|\nabla w\|_2^2\right)\rho^{-\frac{5}{2}}\le\frac{4\|\nabla w\|_2^2 g_0^{3/2}}{3\pi^2}-\frac{8g_0^{5/2}}{5\pi^2}. \tag{4.8}$$

Lemma 4.2 *The expectation value of H_{S1} is bounded above by,*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_{S1}\rangle_\Psi-\rho_0^2V_0\right)\rho^{-5/2}\le\frac{4V_0g_0^{3/2}}{3\pi^2}. \tag{4.9}$$

Lemma 4.3 *The expectation value of H_{S2} is bounded above by,*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_{S2}\rangle_\Psi+2\rho_0^2\|Vw\|_1\right)\rho^{-5/2}\le\frac{2V_0g_0^{3/2}}{\pi^2}. \tag{4.10}$$

Lemma 4.4 *The expectation value of H_{S3} is bounded above by,*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_{S3}\rangle_\Psi - \rho_0^2\|Vw^2\|_1\right)\rho^{-5/2} \le \frac{-2\|Vw\|_1 g_0^{3/2}}{\pi^2}. \tag{4.11}$$

Lemma 4.5 *The expectation value of H_{A1} is bounded above by,*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_{A1}\rangle_\Psi\right)\rho^{-5/2} \le \frac{-8\|Vw\|_1 g_0^{3/2}}{3\pi^2}. \tag{4.12}$$

Lemma 4.6 *The expectation value of H_{A2} is bounded above by,*

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_{A2}\rangle_\Psi\right)\rho^{-5/2} \le \frac{4\|Vw^2\|_1 g_0^{3/2}}{3\pi^2}. \tag{4.13}$$

By definitions of g_0 and w (3.11), (3.12), we have

$$\|\nabla w\|_2^2 - \|Vw\|_1 + \|Vw^2\|_1 = 0, \qquad V_0 - \|Vw\|_1 = g_0. \tag{4.14}$$

Summing (4.8)–(4.13), we have

$$\overline{\lim_{k_c,\rho}}\left(\frac{1}{|\Lambda|}\langle H_N\rangle_\Psi - \rho_0^2 g_0\right)\rho^{-5/2} \le \frac{26 g_0^{5/2}}{15\pi^2}. \tag{4.15}$$

By definition of ρ_0 (3.14), we have proved (4.1). □

5 Estimates on the Numbers of Particles

The first step to prove the Lemma 4.1 to Lemma 4.6 is to estimate the number of particles in the condensate, P_L, P_I, and P_H. This is the main task of this section and we start with the following notations.

Definition 5.1 *Suppose $u_i, k_j \in P$ for $i = 1, \dots, t$, $j = 1, \dots, s$.*

1. *The expectation of the product of particle numbers with momenta $u_1, \dots, u_s$:*

$$Q_\Psi(u_1, u_2, \dots, u_s) = \left\langle \prod_{i=1}^{s} a_{u_i}^\dagger a_{u_i} \right\rangle_\Psi = \sum_{\alpha\in M}\prod_{i=1}^{s}\alpha(u_i)|f(\alpha)|^2.$$

2. *The probability to have m_i particles with momentum u_i, $i = 1, \dots, s$:*

$$Q_\Psi(\{u_1, m_1\}, \dots, \{u_t, m_t\}) \equiv \sum_{\alpha\in A}|f(\alpha)|^2. \tag{5.1}$$

Here $A = \left\{\alpha \in M | \alpha(u_1) = m_1, \dots, \alpha(u_t) = m_t\right\}$.

3. *The expectation of the product of particle numbers with momenta $k_1, \dots, k_s$, conditioned that there are m_i particles with momentum u_i:*

$$Q_\Psi(k_1, \dots, k_s \mid \{u_1, m_1\}, \dots, \{u_t, m_t\})$$

$$\equiv \left(\sum_{\alpha\in A}\prod_{i=1}^{s}\alpha(k_i)|f(\alpha)|^2\right)\left(\sum_{\alpha\in A}|f(\alpha)|^2\right)^{-1},$$

where A is the same as in item 2.

The following theorem provides the main estimates on the number of particles.

Theorem 5.1 *In the limit* $\lim_{k_c\to\infty}\lim_{\rho\to 0}$, $Q_\Psi(u)$ *can be estimated as follows*

$$\lim_{k_c\to\infty}\lim_{\rho\to 0}\left(\rho^{-3/2}|\Lambda|^{-1}\sum_{u\in P_I\cup P_H}Q_\Psi(u)\right)=0, \tag{5.2}$$

$$\lim_{k_c\to\infty}\lim_{\rho\to 0}\left(\rho^{-3/2}|\Lambda|^{-1}\sum_{u\in P_L}Q_\Psi(u)\right)=\frac{1}{3\pi^2}g_0^{3/2}. \tag{5.3}$$

We first collect a few obvious identities of f into the following lemma.

Lemma 5.1

1. *If* $k\in P_I\cup P_H$ *and* $\alpha, \mathcal{A}^k\alpha\in M$, *then*

$$f(\mathcal{A}^k\alpha)=\sqrt{\frac{\alpha(0)}{|\Lambda|}}\sqrt{\frac{\alpha(0)-1}{|\Lambda|}}\lambda_k f(\alpha). \tag{5.4}$$

2. *If* $k\in P_L$, $\alpha\in M_k^s$ *and* $\alpha, \mathcal{A}^k\alpha\in M$, *then*

$$f(\mathcal{A}^k\alpha)=\sqrt{\frac{\alpha(0)}{|\Lambda|}}\sqrt{\frac{\alpha(0)-1}{|\Lambda|}}\lambda_k f(\alpha). \tag{5.5}$$

3. *If* $k\in P_L$, $\alpha\in M_k^a$ *and* $\alpha, \mathcal{A}^k\alpha\in M$, *then*

$$f(\mathcal{A}^k\alpha)=\sqrt{\frac{\alpha(0)}{|\Lambda|}}\sqrt{\frac{\alpha(0)-1}{|\Lambda|}}\sqrt{\frac{\alpha^*(k)+1}{\alpha^*(k)}}\lambda_k f(\alpha). \tag{5.6}$$

4. *If* $\alpha\in M_u^s$ *and* $\mathcal{A}^{u,k}\alpha\in M$, *then*

$$f(\mathcal{A}^{u,k}\alpha)=2\sqrt{\frac{\alpha(0)}{|\Lambda|}}\sqrt{\frac{\alpha(u)}{|\Lambda|}}\sqrt{\lambda_{k+\frac{u}{2}}}\sqrt{\lambda_{-k+\frac{u}{2}}}f(\alpha). \tag{5.7}$$

5. *If* $\alpha\in M_u^a$ *and* $\mathcal{A}^{u,k}\alpha\in M$, *then*

$$f(\mathcal{A}^{u,k}\alpha)=\frac{1}{2\lambda_u}\sqrt{\frac{\alpha(0)}{|\Lambda|}}\sqrt{\frac{|\Lambda|}{\alpha(u)}}\sqrt{\lambda_{k+\frac{u}{2}}}\sqrt{\lambda_{-k+\frac{u}{2}}}f(\alpha). \tag{5.8}$$

In defining the space M, the operation $\mathcal{A}^{u,k}\alpha$ is not allowed when $\alpha\in M_u^a$. However, it is possible through rare coincidences that $\mathcal{A}^{u,k}\alpha\in M$ even if $\alpha\in M_u^a$. Clearly, $\alpha\in M_u^a$ and $\mathcal{A}^{u,k}\alpha\in M$ imply that $\alpha(u)=\alpha(-u)+1$. The following lemma summarizes some properties we need for λ.

Lemma 5.2

1. *For any* $k \in P_L \cup P_I \cup P_H$, λ_k *only depends on* $|k|$ *and*

$$|\lambda_k| \le g_k |k|^{-2} \le g_0 |k|^{-2}, \qquad |\rho\lambda_k| \le 1 - \text{const.}\, \varepsilon_L. \tag{5.9}$$

2. *For any* $k \in P_L$, λ_k *is negative and*

$$-\frac{g_0}{2}\eta_L^2 \rho^{-1} \ge \lambda_u \ge -\rho^{-1}. \tag{5.10}$$

3. *For any* $k \in P_H$, $|\lambda_k|$ *is bounded as*

$$|\lambda_u| \le g_0 \varepsilon_H^{-2}. \tag{5.11}$$

To prove Theorem 5.1, we start with the following estimate on the condensate.

Lemma 5.3 *For any* $\varepsilon > 0$, *when* ρ *is small enough*, *the expected number of zero-momentum particles can be estimated by*

$$|\Lambda|\rho_{-\varepsilon} \le Q_\Psi(0) \le |\Lambda|\rho_\varepsilon. \tag{5.12}$$

5.1 A Lower Bound on the Number of Condensates

Since the total number of particles in fixed to be N, upper bound on $Q_\Psi(u)$ for $(u \ne 0)$ yields a lower bound for $Q_\Psi(0)$. The following lemma provides the upper bounds for expected number of particles in various momentum space regions.

Lemma 5.4 *For small enough* ρ, *the following upper bounds on* $Q_\Psi(u)$ *hold*:

1. *For* $u \in P_I$,

$$Q_\Psi(u) \le \frac{\lambda_u^2\rho^2}{1-\lambda_u^2\rho^2} = \sum_{i=1}^{\infty} (\lambda_u\rho)^{2i}. \tag{5.13}$$

2. *For* $u \in P_L$,

$$Q_\Psi(u) \le \frac{\lambda_u^2\rho^2}{1-\lambda_u^2\rho^2}\left(1 + \text{const.}\, \frac{\rho m_c}{\varepsilon_H}\right). \tag{5.14}$$

3. *For* $u \in P_H$,

$$Q_\Psi(u) \le \text{const.}\, \rho^2 |u|^{-2} |\lambda_u|. \tag{5.15}$$

Proof The basic idea to prove Lemma 5.4 is the following lemma which compares, in particular, $Q_\Psi(\{u, m\})$ and $Q_\Psi(\{u, m-1\})$.

Proposition 5.1 *When* ρ *is small enough*, *for any* $u \in P_I$, *we have*

$$Q_\Psi(\{u, m\}) \le (\lambda_u\rho)^{2i} Q_\Psi(\{u, m-i\}) \quad \text{for } m \ge i \ge 1. \tag{5.16}$$

Proof We start with the following simple observation, whose proof is obvious and we omit it.

Proposition 5.2 *For any $u \in P_I$ fixed and all $\alpha \in M$ with $\alpha(u) = m \geq 1$, there exists a $\beta \in M$ such that $\mathcal{A}^u \beta = \alpha$ and $\beta(u) = m - 1$.*

From the property of f in (5.4) and $\beta(0) \leq N$, we obtain

$$|f(\mathcal{A}^u \beta)| = |\lambda_u| \frac{\beta(0)}{|\Lambda|} |f(\beta)| \leq |\lambda_u| \rho |f(\beta)|.$$

Therefore, we have for $m \geq 1$

$$\begin{aligned} Q_\Psi(\{u, m\}) &\leq \sum_{\beta(u)=m-1} |f(\mathcal{A}^u \beta)|^2 \leq \lambda_u^2 \rho^2 \sum_{\beta(u)=m-1} |f(\beta)|^2 \\ &= \lambda_u^2 \rho^2 Q_\Psi(\{u, m-1\}). \end{aligned} \tag{5.17}$$

This proves (5.16) for $i = 1$. The general cases follow from iterations. □

Together with $\sum_{m=0}^N Q_\Psi(\{u, m\}) = 1$, we have

$$\begin{aligned} Q_\Psi(u) &= \sum_{m=1}^N m Q_\Psi(\{u, m\}) = \sum_{i=1}^N \left(\sum_{m=i}^N Q_\Psi(\{u, m\}) \right) \\ &\leq \sum_{i=1}^N (\lambda_u \rho)^{2i} \left(\sum_{m=0}^N Q_\Psi(\{u, m\}) \right) = \frac{\lambda_u^2 \rho^2}{1 - \lambda_u^2 \rho^2}. \end{aligned} \tag{5.18}$$

This proves (5.13).

We now prove (5.14). Recall that ρ is small, $1 \leq m \leq m_c$ and $u \in P_L$. From the definition of M (3.9), all elements in the asymmetric part, M_u^a, are generated from the symmetric part M_u^s via soft pair creations. Thus

$$\sum_{\alpha:\alpha \in M_u^a}^{\alpha^*(u)=m} |f(\alpha)|^2 \leq \sum_{\beta:\beta \in M_u^s}^{\beta(u)=m} \left(\sum_{k:\pm k + u/2 \in P_H} |f(\mathcal{A}^{u,k} \beta)|^2 \right). \tag{5.19}$$

From (5.7), we have, for $\beta(u) \leq m$,

$$\begin{aligned} |f(\mathcal{A}^{u,k} \beta)|^2 &= 4 |\lambda_{k+u/2} \lambda_{-k+u/2}| \frac{\beta(0)}{|\Lambda|} \frac{\beta(u)}{|\Lambda|} |f(\beta)|^2 \\ &\leq 4 |\lambda_{k+u/2} \lambda_{-k+u/2}| \frac{\rho m}{|\Lambda|} |f(\beta)|^2. \end{aligned} \tag{5.20}$$

Using the upper bound of λ_k in (5.9) and $|u| \ll |k|$, we have

$$\sum_{k:\pm k + u/2 \in P_H} \left| \lambda_{k+u/2} \lambda_{-k+u/2} \right| \leq \sum_{p \in P_H} \text{const.} |p|^{-4} \leq \text{const.} \varepsilon_H^{-1} |\Lambda|. \tag{5.21}$$

Inserting these results into (5.19), we obtain

$$\sum_{\alpha:\alpha \in M_u^a, \alpha^*(u)=m} |f(\alpha)|^2 \leq \text{const.} \frac{\rho m}{\varepsilon_H} \sum_{\beta:\beta \in M_u^s, \beta(u)=m} |f(\beta)|^2. \tag{5.22}$$

Summing the last bound over $1 \le m \le m_c$, we have, for each u fixed,

$$\sum_{\alpha:\alpha\in M_u^a} |f(\alpha)|^2 \le \text{const.}\, \frac{\rho m_c}{\varepsilon_H}. \tag{5.23}$$

Using this method, we can also prove, for $u \ne \pm v$,

$$\sum_{\alpha:\alpha\in M_u^a,\alpha\in M_v^a} |f(\alpha)|^2 \le \text{const.}\left(\frac{\rho m_c}{\varepsilon_H}\right)^2. \tag{5.24}$$

From (5.22), we have, for ρ is small enough

$$Q_\Psi(u) \le \sum_{m=1}^{m_c}\left(m \sum_{\alpha:\alpha\in M_u^s}^{\alpha(u)=m} |f(\alpha)|^2\right)\left(1 + \text{const.}\, \frac{\rho m_c}{\varepsilon_H}\right). \tag{5.25}$$

Following the proof of (5.17), we have the bound

$$\left[\sum_{\alpha:\alpha(u)=m}^{\alpha\in M_u^s} |f(\alpha)|^2\right] \le \lambda_u^2 \rho^2 \left[\sum_{\beta:\beta(u)=m-1}^{\beta\in M_u^s} |f(\beta)|^2\right]. \tag{5.26}$$

Therefore, we can prove (5.14) using the argument of (5.18).

We now prove (5.15) by starting with the following proposition. Once again, the proof is straightforward and we omit it.

Proposition 5.3 *For any $u \in P_H$ fixed and all $\alpha \in M$ with $\alpha(u) = m \ge 1$, either there exists $\beta \in M$ such that $\mathcal{A}^u\beta = \alpha$ and $\beta(u) = m-1$ or there exists $v \in P_L$ and $\beta \in M_v^s$ such that $\alpha = \mathcal{A}^{v,u-v/2}\beta$.*

From this proposition, we have

$$Q_\Psi(\{u,m\}) \le \sum_{\beta:\beta(u)=m-1}\left[|f(\mathcal{A}^u\beta)|^2 + \sum_{v\in P_L,\mathcal{A}^{v,u-v/2}\beta\in M}^{\beta\in M_v^s} |f(\mathcal{A}^{v,u-v/2}\beta)|^2\right]. \tag{5.27}$$

By the properties of f in (5.4), (5.7), we obtain

$$|f(\mathcal{A}^u\beta)|^2 + \sum_{v\in P_L} |f(\mathcal{A}^{v,u-v/2}\beta)|^2 \le \left(\rho^2\lambda_u^2 + \sum_{v\in P_L} 4\rho\frac{\beta(v)}{|\Lambda|}\,|\lambda_u\lambda_{-u+v}|\right)|f(\beta)|^2.$$

Since $v \in P_L$ and $u \in P_H$, from (5.9) we have $|\lambda_u|$, $|\lambda_{-u+v}| \le \text{const.}\,|u|^{-2}$. By definition of M, $\beta(v) \le m_c$. Thus

$$\sum_{v\in P_L} 4\rho\frac{\beta(v)}{|\Lambda|} \le \sum_{v\in P_L} 4\rho\frac{m_c}{|\Lambda|} \le \text{const.}\,\eta_L^{-3} m_c \rho^{5/2}.$$

Hence we have

$$|f(\mathcal{A}^u\beta)|^2 + \sum_{v\in P_L} |f(\mathcal{A}^{v,u-v/2}\beta)|^2 \le \text{const.}\,|u|^{-2}\,|\lambda_u|\,\rho^2 |f(\beta)|^2.$$

Together with the bound in (5.27), we obtain

$$Q_\Psi(\{u,m\}) \le \text{const.}\, |\lambda_u||u|^{-2}\rho^2 Q_\Psi(\{u,m-1\}) \quad \text{for } m \ge 1. \tag{5.28}$$

Summing the last inequality over m, we have proved (5.15). □

The summations in the inequalities in Lemma 5.4 can be performed; we summarize the conclusions in the following lemma.

Proposition 5.4 *Recall that $\varepsilon_L, \eta_L, \varepsilon_H$ are chosen in Definition 3.1 as ρ^η. Then for any k_c and small enough ρ we have*

$$|\Lambda|^{-1}\sum_{u\in P_I} Q_\Psi(u) \le \text{const.}\, \rho^{3/2+\eta}, \tag{5.29}$$

$$|\Lambda|^{-1}\sum_{u\in P_H} Q_\Psi(u) \le \rho^{7/4}, \tag{5.30}$$

$$|\Lambda|^{-1}\sum_{u\in P_L} Q_\Psi(u) \le \left(\frac{g_0^{3/2}}{3\pi^2} + \text{const.}\, \rho^\eta\right)\rho^{3/2}. \tag{5.31}$$

Assuming this proposition, we have, for any $\varepsilon > 0$, when ρ is small enough,

$$Q_\Psi(0) = N - \sum_{u\ne 0} Q_\Psi(u) \ge \rho_{-\varepsilon}|\Lambda|. \tag{5.32}$$

This proves the lower bound in Lemma 5.3. We now prove Proposition 5.4.

Proof The upper bound (5.30) follows from (5.15), $|\lambda_u| \le g_0|u|^{-2}$ (5.9) and the assumption $u \ge \varepsilon_H$ for $u \in P_H$.

To prove the other bounds, we first sum over $u \in P_L$ in (5.14) to have

$$|\Lambda|^{-1}\sum_{u\in P_L} Q_\Psi(u) \le |\Lambda|^{-1}\sum_{u\in P_L} \frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}(1+\rho^{3/4}), \tag{5.33}$$

where we have bounded the factor $\rho m_c/\varepsilon_H$ in the error term by $\rho^{3/4}$.

Let $h(k) = \sqrt{1+4g_0|k|^{-2}}$ and we can rewrite λ as

$$\rho\lambda_{\sqrt{\rho}k} = \frac{1-h(k)}{1+h(k)}. \tag{5.34}$$

Recall for any continuous function F on $\mathbb{R}^3$, we have

$$\frac{1}{L^d}\sum_{p\in\Lambda^*} F(p) = \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*} F(p) \to \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} F(p).$$

Thus we have

$$\lim_{\rho\to 0} |\Lambda|^{-1}\rho^{-3/2}\left(\sum_{u\in P_L} \frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}\right)$$

$$= \lim_{\rho\to 0} \frac{1}{(2\pi)^3} \int_{\varepsilon_L \le |k| \le \eta_L^{-1}} \frac{(h(k)-1)^2}{4h(k)} dk^3 + O(|\Lambda|^{-1/3}). \tag{5.35}$$

The last error comes from replacing the summation by integral.

Due to the choices of ε_L, η_L, we can continue the computation as

$$\lim_{\rho\to 0}\left(\frac{1}{(2\pi)^3} \int_{\varepsilon_L \le |k| \le \eta_L^{-1}} \frac{(h(k)-1)^2}{4h(k)} dk^3\right) + O(|\Lambda|^{-1/3})$$
$$= \frac{1}{3\pi^2} g_0^{3/2} + O(\rho^\eta) + O(|\Lambda|^{-1/3}). \tag{5.36}$$

This proves (5.31) since $L = \rho^{-25/24}$.

Similarly, for $u \in P_I$, we have

$$\lim_{\rho\to 0} |\Lambda|^{-1} \rho^{-3/2} \left(\sum_{u\in P_I} \frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}\right)$$
$$\le \lim_{\rho\to 0} \frac{1}{(2\pi)^3} \int_{\eta_L^{-1} \le |k| \le \infty} \frac{(h(k)-1)^2}{4h(k)} dk^3 + O(|\Lambda|^{-1/3}). \tag{5.37}$$

This proves (5.29) and concludes Proposition 5.4. □

As a corollary to the proof, we have the following estimates.

Corollary 5.1

$$\lim_{n\to\infty} \lim_{\rho\to 0} |\Lambda|^{-1} \rho^{-3/2} \left(\sum_{u\in P_L} \sum_{m=0}^{n} (\rho\lambda_u)^{2m}\right) = \frac{g_0^{3/2}}{3\pi^2}. \tag{5.38}$$

Proof From the previous proof, we only need to prove the tail terms vanishes. Recall $|\rho\lambda_u| \le 1 - \text{const.}\, \varepsilon_L < 1$ in (5.9). Thus we have

$$\lim_{n\to\infty} \lim_{\rho\to 0} |\Lambda|^{-1} \rho^{-3/2} \left(\sum_{u\in P_L} \sum_{m=n+1}^{\infty} (\rho\lambda_u)^{2m}\right)$$
$$\le \lim_{n\to\infty} \lim_{\rho\to 0} |\Lambda|^{-1} \rho^{-3/2} \left(\sum_{u\in\Lambda^*} \frac{(\rho\lambda_u)^{2n+2}}{1-(\rho\lambda_u)^2}\right)$$
$$\le \lim_{\rho\to 0} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} H(2n) dk^3 + O(|\Lambda|^{-1/3}), \tag{5.39}$$

where

$$H(2n) = \frac{(h(k)-1)^2 (\frac{1-h(k)}{1+h(k)})^{2n}}{4h(k)}.$$

By Lebesgue monotone convergence theorem, we have that $H(2n)$ converges to zero. This proves the corollary. □

We note that (5.28) also shows that, for $u \in P_H$, $Q_\Psi(\{u,m\})$ is exponentially small with m, i.e.,

$$Q_\Psi(\{u,m\}) \le (\text{const. } |\lambda_u|\, \rho^2 |u|^{-2})^m. \tag{5.40}$$

Furthermore, using similar method, one can easily generalize this result to: for $u, v \in P_H$ and $u+v \neq 0$

$$Q_\Psi(\{u,m\},\{v,n\}) \le \left(\text{const. } |\lambda_u| \rho^2 \varepsilon_H^{-2}\right)^m \left(\text{const. } |\lambda_v| \rho^2 \varepsilon_H^{-2}\right)^n, \tag{5.41}$$

which implies, for $u, v \in P_H$ and $u+v \neq 0$, the following inequality:

$$Q_\Psi(u,v) \le \text{const. } |\lambda_u \lambda_v|\, \rho^4 \varepsilon_H^{-4}. \tag{5.42}$$

5.2 Proof of Lemma 5.3: Upper Bound

Proposition 5.4 states that the density of particles with momenta in P_I and P_H are much smaller than $\rho^{3/2}$. And it implies an upper bound on the density of particles with momenta in P_L. We now prove a matching lower bound

$$\sum_{u \in P_L} Q_\Psi(u) \ge \left(\frac{1}{3\pi^2} g_0^{3/2} - \varepsilon\right) \rho^{3/2} \Lambda \tag{5.43}$$

for ρ small enough. Since the total number of particles is fixed, this will provide a upper bound on the number of particles in the condensate and hence proves the upper bound part of Lemma 5.3.

We start with the following lemma, which bounds the average number of particles in the condensate under the condition that there are at most k particles with momentum u.

Proposition 5.5 *For $u \in P_I$ and for any k fixed with $0 \le k \le m_c$ (m_c defined in (3.8)), we have, for ρ small enough,*

$$\frac{\sum_{i=0}^k Q_\Psi(0|\{u,i\}) Q_\Psi(\{u,i\})}{\sum_{i=0}^k Q_\Psi(\{u,i\})} \ge N - \text{const. } N \rho^{1/2} m_c. \tag{5.44}$$

Proof By (5.22), the contribution of $\alpha \in M_u^a$ to $Q_\Psi(\{u,m\})$ for $1 \le m \le m_c$ is of lower order when compared with the contribution of $\alpha \in M_u^s$. The ratio of the contributions from $\alpha \in M_u^s$ between $Q_\Psi(\{u,m\})$ and $Q_\Psi(\{u,m-1\})$ is estimated in (5.26). Together with the upper bound on $|\lambda_u|$ in (5.9) and the choices of $\varepsilon_L, \varepsilon_H$, we have for ρ small enough,

$$\frac{Q_\Psi(\{u,m\})}{Q_\Psi(\{u,m-1\})} \le \left(\rho^2 \lambda_u^2\right)\left(1 + \text{const. } \frac{m_c \rho}{\varepsilon_H}\right) \le \left(1 - \text{const. } \left(\varepsilon_L - \frac{m_c \rho}{\varepsilon_H}\right)\right) < 1. \tag{5.45}$$

Hence $Q_\Psi(\{u,m\})$ is monotonic decrease in m. We thus have for $0 \le k \le m_c$,

$$\sum_{i=0}^k Q_\Psi(\{u,i\}) \ge \frac{k+1}{m_c+1} \sum_{i=0}^{m_c} Q_\Psi(\{u,i\}) = \frac{k+1}{m_c+1}, \tag{5.46}$$

where the last identity is the normalization of the state Ψ.

By definition of $Q_\Psi(0|\{u,i\})$ and (5.32), we have

$$\sum_{i=0}^{m_c} Q_\Psi(0|\{u,i\})Q_\Psi(\{u,i\}) = Q_\Psi(0) \geq N - \text{const.}\, N\rho^{1/2}.$$

On the other hand, for any m, $Q_\Psi(0|\{u,m\}) \leq N$. Hence, the numerator on the left side of (5.44) can be bounded by:

$$\begin{aligned}
&\sum_{i=0}^{k} Q_\Psi(0|\{u,i\})Q_\Psi(\{u,i\}) \\
&= \sum_{i=0}^{m_c} Q_\Psi(0|\{u,i\})Q_\Psi(\{u,i\}) - \sum_{i=k+1}^{m_c} Q_\Psi(0|\{u,i\})Q_\Psi(\{u,i\}) \\
&\geq N - \text{const.}\, N\rho^{1/2} - N\sum_{i=k+1}^{m_c} Q_\Psi(\{u,i\}) \\
&= N\sum_{i=0}^{k} Q_\Psi(\{u,i\}) - \text{const.}\, N\rho^{1/2},
\end{aligned} \tag{5.47}$$

where we have used $\sum_{i=0}^{m_c} Q_\Psi(\{u,i\}) = 1$ in the last identity. Finally, we divide (5.47) by $\sum_{i=0}^{k} Q_\Psi(\{u,i\})$ and use (5.46) to conclude (5.44). □

Return to the proof of (5.43) for $u \in P_L$. Since $\mathcal{A}^u\beta$ is a one to one map (not necessarily surjective), we have

$$\sum_{i=1}^{m_c} Q_\Psi(\{u,i\}) \geq \sum_{\beta(u)=0}^{m_c-1} |f(\mathcal{A}^u\beta)|^2. \tag{5.48}$$

From (5.5) and (5.6), the right hand side is bounded below by

$$\lambda_u^2|\Lambda|^{-2}\sum_{\beta(u)=0}^{m_c-1}(\beta(0)^2-\beta(0))|f(\beta)|^2. \tag{5.49}$$

By Jensen's inequality and $\beta(0) \leq N$, it is bounded below by

$$\lambda_u^2|\Lambda|^{-2}\left(\left(\frac{\sum_{\beta(u)=0}^{m_c-1}\beta(0)|f(\beta)|^2}{\sum_{\beta(u)=0}^{m_c-1}|f(\beta)|^2}\right)^2 - N\right)\sum_{\beta(u)=0}^{m_c-1}|f(\beta)|^2. \tag{5.50}$$

By definition,

$$\frac{\sum_{\beta(u)=0}^{m_c-1}\beta(0)|f(\beta)|^2}{\sum_{\beta(u)=0}^{m_c-1}|f(\beta)|^2} = \frac{\sum_{i=0}^{m_c-1} Q_\Psi(0|\{u,i\})Q_\Psi(\{u,i\})}{\sum_{i=0}^{m_c-1} Q_\Psi(\{u,i\})}. \tag{5.51}$$

The term on the right hand side can be estimated by Proposition 5.5. Combining all estimates up to now and we obtain

$$\sum_{i=1}^{m_c} Q_\Psi(\{u,i\}) \geq ((\rho-\rho^{5/4})\lambda_u)^2\sum_{i=0}^{m_c-1} Q_\Psi(\{u,i\}). \tag{5.52}$$

Finally, using (5.46), we have

$$\sum_{i=1}^{m_c} Q_\Psi(\{u,i\}) \ge ((\rho-\rho^{5/4})\lambda_u)^2\left(1-\frac{1}{m_c+1}\right). \tag{5.53}$$

We can generalize this result as follows. For $m \ge 1$, we first iterate the argument in proving (5.48) and (5.49) to have

$$\lambda_u^{2m}|\Lambda|^{-2m}\sum_{\beta(u)=0}^{m_c-m}(\beta(0)-2m)^{2m}|f(\beta)|^2 \le \sum_{i=m}^{m_c} Q_\Psi(\{u,i\}). \tag{5.54}$$

Again, using Jensen's inequality, Proposition 5.5, and (5.46), we have

$$\sum_{i=m}^{m_c} Q_\Psi(\{u,i\} \ge ((\rho-\rho^{5/4})\lambda_u)^{2m}\left(1-\frac{m}{m_c+1}\right). \tag{5.55}$$

So with the fact $m_c=\rho^{-\eta}$, $Q_\Psi(u)$ can be bounded as follows,

$$\begin{aligned} Q_\Psi(u) = \sum_{m=1}^{\infty}\sum_{i=m}^{\infty} Q_\Psi(\{u,i\}) &\ge \sum_{m=1}^{m_c}((\rho-\rho^{5/4})\lambda_u)^{2m}\left(1-\frac{m}{m_c+1}\right) \\ &\ge (1-\rho^{\eta/2})\sum_{i=1}^{\sqrt{m_c}+1}(\rho\lambda_u)^{2i}. \end{aligned} \tag{5.56}$$

Now the summation over $u \in P_L$ was carried out in Corollary 5.38 and we have proved (5.43). Since the total number of particle is N, the bounds on $Q_\Psi(0)$ follows from (5.43) and Proposition 5.4. This concludes Lemma 5.3.

The previous method can be applied to yield the following estimates which will be useful later on.

Lemma 5.5 *For $u \in P_L$ and ρ sufficiently small, the following two bounds hold*:

$$\sum_{m=m_c-1}^{m_c} mQ_\Psi(\{u,m\}) \le \frac{\rho^2\lambda_u^2}{1-\rho^2\lambda_u^2}\rho^{\eta/2}, \tag{5.57}$$

$$\sum_{\alpha(u)\le m_c-2}|f(\alpha)|^2\alpha(0)^2\alpha(u) \ge N^2\frac{\rho^2\lambda_u^2}{1-\rho^2\lambda_u^2}(1-2\rho^{\eta/2}-(\rho\lambda_u)^{2\sqrt{m_c}}). \tag{5.58}$$

Proof Because $Q_\Psi(\{u,m\})$ is monotonic decrease in m, we have

$$\sum_{m=m_c-1}^{m_c} mQ_\Psi(\{u,m\}) \le \frac{\text{const.}}{m_c}\sum_{m=1}^{m_c} mQ_\Psi(\{u,m\}) = \frac{\text{const.}}{m_c}Q_\Psi(u). \tag{5.59}$$

Together with the upper bound (5.13) on $Q_\Psi(u)$, we have proved (5.57).

To prove (5.58), we follow the argument in (5.54) to have, for $m \le m_c-2$,

$$\lambda_u^{2m}|\Lambda|^{-2m}\sum_{\beta(u)=0}^{m_c-m-2}(\beta(0)-2m)^{2m+2}|f(\beta)|^2 \le \sum_{i=m}^{m_c-2} Q_\Psi(0,0|\{u,i\})Q_\Psi(\{u,i\}).$$

Again, using Jensen's inequality, Proposition 5.5 and (5.46), we have

$$\sum_{\alpha(u)\leq m_c-2} |f(\alpha)|^2\alpha(0)^2\alpha(u) = \sum_{m=0}^{m_c-2}\sum_{i=m}^{m_c-2} Q_\Psi(0,0|\{u,i\})Q_\Psi(\{u,i\})$$

$$\geq (1-2\rho^{\eta/2})\sum_{i=1}^{\sqrt{m_c}}(\rho\lambda_u)^{2i}N^2. \tag{5.60}$$

This implies (5.58). □

Lemma 5.3 can be extended to the following estimate:

Lemma 5.6 *With the assumptions in Lemma 5.3, $Q_\Psi(0,0)$ satisfies the estimate*

$$(\Lambda\rho_{-\varepsilon})^2 \leq Q_\Psi(0,0) \leq (\Lambda\rho_\varepsilon)^2. \tag{5.61}$$

Proof By Jensen's inequality and Lemma 5.3, we have the lower bound

$$Q_\Psi(0,0) \geq [Q_\Psi(0)]^2 \geq (\Lambda\rho_{-\varepsilon})^2.$$

For the upper bound, we start with

$$Q_\Psi(0,0) = N^2 - 2N\sum_{u\neq 0} Q_\Psi(u) + \sum_{u,v\neq 0} Q_\Psi(u,v)$$

$$\leq (Q_\Psi(0))^2 + \sum_{u,v\neq 0} Q_\Psi(u,v). \tag{5.62}$$

Since the number of particles with momentum $u \in P_L$ is at most m_c,

$$\sum_{u\in P_L, v\neq 0} Q_\Psi(u,v) \leq \left(\sum_{u\in P_L} m_c\right)\left(\sum_{v\neq 0} Q_\Psi(v)\right). \tag{5.63}$$

By definition of P_L, we have $\sum_{u\in P_L} m_c = m_c\eta_L^{-3}\rho^{3/2}\Lambda$. The last factor in (5.63) can be estimated by Proposition 5.4. Thus we have

$$\sum_{u\in P_L, v\neq 0} Q_\Psi(u,v) = o(\rho^{5/2}|\Lambda|^2). \tag{5.64}$$

For the terms $\sum_{u\in P_I\cup P_H, v\neq 0}$, the upper bound on the total number of particles in P_I and P_H in Proposition 5.4 yields that

$$\sum_{u\in P_I\cup P_H, v\neq 0} Q_\Psi(u,v) \leq \sum_{u\in P_I\cup P_H} Q_\Psi(u)N = o(\rho^{5/2}|\Lambda|^2). \tag{5.65}$$

Inserting (5.64), (5.65) into (5.62) and using the upper bound in Lemma 5.3, we obtain the upper bound on $Q_\Psi(0,0)$. □

6 Estimates on Kinetic Energy

In this section, we will prove the kinetic energy estimate Lemma 4.1. This lemma follows immediately from summing the estimates ((6.2)–(6.4)) of the next lemma.

Lemma 6.1 *In the limit $\rho \to 0$, $Q_\Psi(u, v)$ can be bounded above by*

$$\lim_{\rho\to 0}\left(\rho^{-5/2}|\Lambda|^{-2}\sum_{u,v\neq 0} Q_\Psi(u,v)\right)\le 0. \tag{6.1}$$

Furthermore, $\sum u^2 Q_\Psi(u)$ can be bounded above as follows

$$\overline{\lim_{\rho\to 0}}\left(\rho^{-5/2}|\Lambda|^{-1}\sum_{u\in P_I} u^2(Q_\Psi(u)-(\rho_0\lambda_u)^2)\right)\le 0, \tag{6.2}$$

$$\overline{\lim_{\rho\to 0}}\left(\rho^{-5/2}|\Lambda|^{-1}\sum_{u\in P_L} u^2(Q_\Psi(u)-(\rho_0 w_u)^2)\right)\le -\frac{8}{5\pi^2}g_0^{5/2}, \tag{6.3}$$

$$\overline{\lim_{\rho\to 0}}\left(\frac{\rho^{-5/2}}{|\Lambda|}\sum_{u\in P_H} u^2\left(Q_\Psi(u)-\left(\rho_0^2+\frac{4g_0^{3/2}}{3\pi^2}\rho_0^{5/2}\right)\lambda_u^2\right)\right)\le 0. \tag{6.4}$$

Proof The bound (6.1) was proved in (5.64) and (5.65). We now prove (6.2) concerning $u \in P_I$.

The upper bound of $Q_\Psi(u)$ in (5.13) can be rewritten as

$$Q_\Psi(u)\le(\rho\lambda_u)^2+\frac{(\rho\lambda_u)^4}{1-(\rho\lambda_u)^2}. \tag{6.5}$$

Recall $\rho_0=\rho(1+O(\sqrt{\rho}))$ and the bounds on λ in (5.9). Since $\rho^{1/2}\ll|u|\ll 1$ when $u\in P_I$, see Definition 3.1, the error term of the last bound can be estimated by

$$\overline{\lim_{\rho\to 0}}\,|\Lambda|^{-1}\rho^{-5/2}\sum_{u:\rho^{1/2}\ll|u|\ll 1} u^2\frac{(\rho\lambda_u)^4}{1-(\rho\lambda_u)^2}=0. \tag{6.6}$$

This proves (6.2).

We now prove (6.3) concerning $u \in P_L$. Following the strategy of the previous argument, we first use $0\ge 1-(\rho_0\lambda_u)^2\ge \text{const.}\,\varepsilon_L$ in (5.9) and (5.10) to rewrite the upper bound of $Q_\Psi(u)$ in (5.14) as

$$Q_\Psi(u)\le\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}+\text{const.}\,\frac{\rho m_c}{\varepsilon_H\varepsilon_L}. \tag{6.7}$$

The error terms are negligible in the sense that

$$\sum_{u\in P_L}u^2\frac{\rho m_c}{\varepsilon_H\varepsilon_L}=o(\rho^{5/2}\Lambda).$$

Since $w_u=g_u|u|^{-2}$, $\rho_0-\rho=O(\rho^{3/2})$ and $|g_u-g_0|\le\text{const.}\,|u|$, we have

$$\lim_{\rho}\sum_{u\in P_L}u^2\left(\left(\frac{\rho g_0}{u^2}\right)^2-(\rho_0 w_u)^2\right)\rho^{-5/2}|\Lambda|^{-1}=0. \tag{6.8}$$

Summarize what we have proved, we have the following inequality:

$$\begin{aligned}
&\overline{\lim_{\rho}} \sum_{u\in P_L} u^2(Q_\Psi(u) - (\rho_0 w_u)^2)\rho^{-5/2}|\Lambda|^{-1} \\
&\quad\le \overline{\lim_{\rho}} \sum_{u\in P_L} u^2\left(\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2} - \left(\frac{\rho g_0}{u^2}\right)^2\right)\rho^{-5/2}|\Lambda|^{-1}.
\end{aligned} \tag{6.9}$$

Let $u = \sqrt{\rho}k$ and $h(k) = \sqrt{1+4g_0|k|^{-2}}$ as in (5.34). Then the right hand side of (6.9) is estimated as

$$\frac{1}{(2\pi)^3}\int_{\varepsilon_L\le|k|\le\eta_L^{-1}} k^2\left(\frac{1+2g_0|k|^{-2}}{2h(k)} - \frac{1+2(g_0|k|^{-2})^2}{2}\right)dk^3 + O(|\Lambda|^{-1/3}).$$

Direct calculation yields that

$$\frac{1}{(2\pi)^3}\int_{k\in\mathbb{R}^3} k^2\left(\frac{1+2g_0|k|^{-2}}{2h(k)} - \frac{1+2(g_0|k|^{-2})^2}{2}\right)dk^3 = -\frac{8}{5\pi^2}g_0^{5/2}. \tag{6.10}$$

Inserting this result into (6.9), we obtain the desired result (6.3).

Finally, we prove (6.4) concerning $u \in P_H$. Recall the bound (5.28) on the ratio of $Q_\Psi(\{u,m\})/Q_\Psi(\{u,m-1\})$. Since $|\lambda_u| \le g_0|u|^{-2}$ (5.9) and $u \in P_H$, the factor on the right hand side of (5.28) can be bounded by $\rho^{3/2}$. Thus we have

$$Q_\Psi(u) = \sum_m mQ_\Psi(\{u,m\}) \le \sum_{m\ge1} Q_\Psi(\{u,m\})(1+O(\rho^{3/2})). \tag{6.11}$$

We now repeat the argument from (5.27) to (5.28) but refine the proof by using Proposition 5.4. Hence for any $u \in P_H$, we have

$$\begin{aligned}
\sum_{m\ge1} Q_\Psi(\{u,m\}) &\le \sum_\beta\left(\frac{\beta(0)^2}{|\Lambda|^2}\lambda_u^2 + \sum_{v\in P_L} 4\frac{\beta(0)}{|\Lambda|}\frac{\beta(v)}{|\Lambda|}|\lambda_u\lambda_{-u+v}|\right)|f(\beta)|^2 \\
&\le |\Lambda|^{-2}\lambda_u^2 Q_\Psi(0,0) + \sum_{v\in P_L}\rho|\Lambda|^{-1}\left(4Q_\Psi(v)\,|\lambda_u\lambda_{-u+v}|\right).
\end{aligned} \tag{6.12}$$

By mean value theorem and $\lambda_k = -g_k|k|^{-2}$ for $k \in P_H$, we have that $\exists\tilde{u}\in\mathbb{R}^3 : |\tilde{u}-u| \le v$ s.t.

$$|\lambda_{-u+v} - \lambda_{-u}| \le \text{const.}\left(\left|\frac{\partial g_{\tilde{u}}}{\partial\tilde{u}}\right|\tilde{u}^{-2} + |g_{\tilde{u}}|\tilde{u}^{-3}\right)|v|. \tag{6.13}$$

From the estimates (5.9) on λ_u and $u \sim \tilde{u}$, we obtain:

$$\begin{aligned}
|\lambda_u||\lambda_{-u+v} - \lambda_{-u}| &\le \text{const.}\left(\left|\frac{\partial g_{\tilde{u}}}{\partial\tilde{u}}g_u\right|u^{-4} + |g_{\tilde{u}}||g_u|u^{-5}\right)|v| \\
&\le \text{const.}\,|u|^{-2}\varepsilon_H^{-3}G(u)|v|,
\end{aligned} \tag{6.14}$$

where by Schwarz inequality, we have:

$$G(u) = \max_{u':|u'-u|\le\eta_L^{-1}\rho^{1/2}}\left\{\left|\frac{\partial g_{u'}}{\partial u'}\right|^2 + |g_{u'}|^2\right\}. \tag{6.15}$$

We note that it is easy to check $\sum_{u\in P_H} G(u)/\Lambda < \infty$. Together with the results on the total number of P_L particles in (5.3), we obtain that, for ρ small enough and $u \in P_H$, the last term in (6.12) is bounded above by

$$\lambda_u^2 \rho^{5/2}\left(\frac{4g_0^{3/2}}{3\pi^2} + \rho^\eta\right) + \frac{\text{const.}\ \rho^3}{u^2 \varepsilon_H^3 \eta_L} G(u). \tag{6.16}$$

The $Q_\psi(0,0)$ in the last second term of (6.12) is bounded by Lemma 5.6. Inserting (6.16) and (6.12) into (6.11) and using $\lambda_u^2 = w_u^2$ for $u \in P_H$, we obtain that,

$$\overline{\lim_{\rho\to 0}} \sum_{u\in P_H} u^2 \left(Q_\Psi(u) - (\rho_0 w_u)^2 \left(1 + \left[\frac{4g_0^{3/2}}{3\pi^2}\right]\rho_0^{1/2}\right)\right) \frac{\rho^{-5/2}}{|\Lambda|} \le 0. \tag{6.17}$$

This proves (6.4). □

7 Estimates on Pair Interaction Energies

7.1 Proof of Lemma 4.2

First, with the fact $a_u^\dagger a_u^\dagger a_u a_u \le (a_u^\dagger a_u)^2$ and $0 \le |V_u| \le V_0$ for any u, we can bound H_{S1} as follows

$$\begin{aligned} H_{S1} &\le V_0 \Lambda^{-1} \sum_{u,v} a_u^\dagger a_u a_v^\dagger a_v + \Lambda^{-1} \sum_{u\neq v} V_{u-v} a_u^\dagger a_u a_v^\dagger a_v \\ &\le V_0 N\rho + V_0 \Lambda^{-1} \sum_{u\neq v} a_u^\dagger a_v^\dagger a_v a_u = 2V_0 N\rho - V_0\Lambda^{-1} \sum_u (a_u^\dagger a_u)^2. \end{aligned} \tag{7.1}$$

Therefore we can bound the expectation value $\langle H_{S1}\rangle$:

$$\langle H_{S1}\rangle_\Psi \le 2V_0 N\rho - V_0 \Lambda^{-1} \sum_u Q_\Psi(u,u) \le 2V_0 N\rho - V_0\Lambda^{-1} Q_\Psi(0,0). \tag{7.2}$$

By the lower bounds of $Q_\Psi(0,0)$ in Lemma 5.6 and the definition of ρ_0 in (3.14), we have proved Lemma 4.2.

7.2 Proof of Lemma 4.3

We start the proof with the following identity for $\langle\Psi|a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4}|\Psi\rangle$.

Lemma 7.1 *For any fixed $u_{1,2,3,4} \in \Lambda^*$ and $\alpha \in M$, define $T(\alpha)$ to be the state*

$$|T(\alpha)\rangle = C a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4} |\alpha\rangle, \tag{7.3}$$

where C is the positive normalization constant when $|T(\alpha)\rangle \neq 0$. Then we have

$$\langle\Psi|a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4}|\Psi\rangle = \sum_{\alpha\in M} f(\alpha)\overline{f(T(\alpha))}\sqrt{\langle\alpha|a_{u_4}^\dagger a_{u_3}^\dagger a_{u_2} a_{u_1} | a_{u_1}^\dagger a_{u_2}^\dagger a_{u_3} a_{u_4}|\alpha\rangle}. \tag{7.4}$$

The map T depends on $u_{1,2,3,4}$ and in principle it has to carry them as subscripts. We omit these subscripts since it will be clear from the context what they are.

Proof For any $u_{1,2,3,4} \in \Lambda^*$ fixed, by definition of Ψ, we have

$$\langle\Psi|a^\dagger_{u_1}a^\dagger_{u_2}a_{u_3}a_{u_4}|\Psi\rangle = \sum_{\alpha,\beta\in M} f(\alpha)\overline{f(\beta)}\langle\beta|a^\dagger_{u_1}a^\dagger_{u_2}a_{u_3}a_{u_4}|\alpha\rangle. \tag{7.5}$$

By definition of M, we have

$$\langle\beta|a^\dagger_{u_1}a^\dagger_{u_2}a_{u_3}a_{u_4}|\alpha\rangle \neq 0 \quad\Rightarrow\quad \beta = T(\alpha). \tag{7.6}$$

Since $|T(\alpha)\rangle$ is normalized, the identity in Lemma 7.1 is obvious. □

Lemma 4.3 follows from the following lemma and $\lambda_u = -w_u$ for $u \in P_H \cup P_I$. Notice that the factor 2 in the estimate of Lemma 4.3 is due to the complex conjugate in the definition of H_{S2}. Similar factor also appears in Lemma 4.5.

Lemma 7.2

$$\overline{\lim_{m_c,\rho}} \sum_{u\in P_I\cup P_H} (\langle V_u|\Lambda|^{-1}a^\dagger_u a^\dagger_{-u}a_0a_0\rangle - \rho_0^2 V_u\lambda_u)\rho^{-5/2}|\Lambda|^{-1} = 0, \tag{7.7}$$

$$\overline{\lim_{m_c,\rho}} \sum_{u\in P_L} (\langle V_u|\Lambda|^{-1}a^\dagger_u a^\dagger_{-u}a_0a_0\rangle + \rho_0^2 V_u w_u)\rho^{-5/2}|\Lambda|^{-1} \le \frac{V_0 g_0^{3/2}}{\pi^2}. \tag{7.8}$$

Proof We first prove (7.7) concerning with $u \in P_I \cup P_H$. By Lemma 7.1, we have

$$\begin{aligned}\langle V_u|\Lambda|^{-1}a^\dagger_u a^\dagger_{-u}a_0a_0\rangle &= V_u|\Lambda|^{-1} \sum_{\alpha:\alpha\in M,\mathcal{A}^u\alpha\in M} f(\alpha)f(\mathcal{A}^u\alpha)\\ &\quad\times\sqrt{(\alpha(0)^2-\alpha(0))(\alpha(u)+1)(\alpha(-u)+1)}.\end{aligned} \tag{7.9}$$

The case that $\alpha \in M$ and $\mathcal{A}^u\alpha \notin M$ can only happen when $\alpha(0) = 0$ or 1 and thus has no contribution. From the relation between $f(\alpha)$ and $f(\mathcal{A}^u\alpha)$ in (5.4), we have

$$(7.9) = \lambda_u V_u|\Lambda|^{-2}\sum_{\alpha\in M}|f(\alpha)|^2\alpha(0)(\alpha(0)-1)\sqrt{(\alpha(u)+1)(\alpha(-u)+1)}. \tag{7.10}$$

By the Schwarz inequality, we have

$$\begin{aligned}&\left|\sum_\alpha \alpha(0)(\alpha(0)-1)\left(\sqrt{(\alpha(u)+1)(\alpha(-u)+1)}-1\right)|f(\alpha)|^2\right|\\ &\quad\le N^2\left|\sum_\alpha\frac{\alpha(u)+\alpha(-u)}{2}|f(\alpha)|^2\right| = N^2 Q_\Psi(u).\end{aligned} \tag{7.11}$$

Inserting (7.11) into (7.10) and summing over $u \in P_I \cup P_H$ of (7.10), we obtain

$$\sum_{u\in P_I\cup P_H}(\langle V_u|\Lambda|^{-1}a^\dagger_u a^\dagger_{-u}a_0a_0\rangle - V_u\lambda_u(Q_\Psi(0,0)-Q_\Psi(0)))$$

$$\leq \text{const.}\, \rho^2 |\Lambda| \sum_{u \in P_I \cup P_H} Q_\Psi(u). \tag{7.12}$$

From the upper bound of $\sum Q_\Psi(u)$ in (5.29), the right hand side of above inequality is bounded by $(o(\rho^{5/2}\Lambda))$. By the bounds on $Q_\Psi(0,0)$ in Lemma 5.6, we have proved (7.7).

To prove (7.8) concerning $u \in P_L$, we note that (7.9) still holds, but $\mathcal{A}^u\alpha \notin M$ when $\alpha^*(u) = m_c$. Therefore, for $u \in P_L$, (7.9) is equal to

$$V_u |\Lambda|^{-1} \sum_{\alpha:\alpha \in M, \alpha^*(u) < m_c} f(\alpha) f(\mathcal{A}^u\alpha) \sqrt{\alpha(0)(\alpha(0)-1)(\alpha(u)+1)(\alpha(-u)+1)}.$$

We can express $f(\mathcal{A}^u\alpha)$ in terms of $f(\alpha)$; in both cases: $\alpha \in M_u^s$ or $\alpha \in M_u^a$, we have the following identity:

$$\begin{aligned} &f(\alpha) f(\mathcal{A}^u\alpha)\sqrt{(\alpha(u)+1)(\alpha(-u)+1)} \\ &\quad = \lambda_u |f(\alpha)|^2 |\Lambda|^{-1} \sqrt{\alpha(0)(\alpha(0)-1)}(\alpha^*(u)+1). \end{aligned} \tag{7.13}$$

Hence, for $u \in P_L$,

$$(7.9) = \sum_{\alpha:\alpha \in M, \alpha^*(u) < m_c} \lambda_u V_u |\Lambda|^{-2} |f(\alpha)|^2 \alpha(0)(\alpha(0)-1)(\alpha^*(u)+1). \tag{7.14}$$

We note $\lambda_u < 0$ and $V_u \approx V_0 > 0$, for $u \in P_L$. For any $\alpha \in M$, $\alpha^*(u) - \alpha(u) \leq 1$ by definition. Hence we can replace the summation $\alpha^*(u) < m_c$ by $\alpha(u) \leq m_c - 2$ to have an upper bound. Summing over $u \in P_L$ of (7.14), we have

$$\begin{aligned} &\left\langle \sum_{u \in P_L} V_u |\Lambda|^{-1} a_u^\dagger a_{-u}^\dagger a_0 a_0 \right\rangle \\ &\quad \leq \sum_{u \in P_L} \sum_{\alpha(u) \leq m_c - 2} \lambda_u V_u |\Lambda|^{-2} |f(\alpha)|^2 \alpha(0)(\alpha(0)-1)\alpha(u) \\ &\qquad + \sum_{u \in P_L} \sum_{\alpha(u) \leq m_c - 2} \lambda_u V_u |\Lambda|^{-2} |f(\alpha)|^2 \alpha(0)(\alpha(0)-1). \end{aligned} \tag{7.15}$$

The last term is equal to

$$\begin{aligned} &\sum_{u \in P_L} \lambda_u V_u |\Lambda|^{-2} (Q_\Psi(0,0) - Q_\Psi(0)) \\ &\quad - \sum_{u \in P_L} \sum_{i=m_c-1}^{m_c} \lambda_u V_u |\Lambda|^{-2} (Q_\Psi(0,0|u,i) Q_\Psi(u,i)). \end{aligned} \tag{7.16}$$

Since $Q_\Psi(0,0|u,i) \leq N^2$, the last term in (7.16) is bounded from above by

$$\sum_{u \in P_L} \sum_{i=m_c-1}^{m_c} \text{const.}\, |\lambda_u \rho^2 Q_\Psi(u,i)| \leq o(\rho^{5/2}\Lambda), \tag{7.17}$$

where we have used (5.45). For the first term of (7.16), we can bound it by using Lemma 5.6. We now use (5.58) to estimate the first term on the right hand side of (7.15). Combining these results, we have

$$\left\langle \sum_{u\in P_L} V_u|\Lambda|^{-1}a_u^\dagger a_{-u}^\dagger a_0a_0\right\rangle \le \sum_{u\in P_L}\lambda_u V_u\rho^2\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}(1-2\rho^{\frac{\eta}{2}}-(\rho\lambda_u)^{2\sqrt{m_c}}) + \sum_{u\in P_L}\lambda_u V_u\rho_0^2+o(\rho^{5/2}\Lambda). \tag{7.18}$$

Since $|\lambda_u\rho|\le 1$ and $|V_u|\le V_0$, we have

$$\sum_{u\in P_L}|\lambda_u V_u|\rho^2\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}\le\sum_{u\in P_L}V_0\rho\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}\le \text{const.}\,\rho^{5/2}\Lambda. \tag{7.19}$$

By (5.39), we have

$$\sum_{u\in P_L}|\lambda_u V_u|\rho^2\frac{(\rho\lambda_u)^2}{1-(\rho\lambda_u)^2}(\rho\lambda_u)^{2\sqrt{m_c}}\le o(\rho^{5/2}\Lambda). \tag{7.20}$$

Inserting (7.19)–(7.20) into (7.18), we have

$$\begin{aligned}&\sum_{u\in P_L}(\langle V_u|\Lambda|^{-1}a_u^\dagger a_{-u}^\dagger a_0a_0\rangle + w_uV_u\rho_0^2)\\ &\quad\le \sum_{u\in P_L}(\lambda_u+w_u)V_u\rho_0^2+\sum_{u\in P_L}V_u\rho^2\frac{\lambda_u^3\rho^2}{1-\rho^2\lambda_u^2}+o(\rho^{5/2}\Lambda).\end{aligned} \tag{7.21}$$

Since $|g_u-g_0|+|V_u-V_0|\le \text{const.}\,|u|$, we can replace w_u and V_u by $g_0|u|^{-2}$ and V_0 in last inequality so that the rhs of (7.21) is bounded by

$$\begin{aligned}&V_0\rho_0^2\sum_{u\in P_L}(\lambda_u+g_0|u|^{-2})+V_0\rho^2\sum_{u\in P_L}\frac{\lambda_u^3\rho^2}{1-\rho^2\lambda_u^2}+o(\rho^{5/2}\Lambda)\\ &\quad= V_0\rho_0^2\sum_{u\in P_L}\left(\lambda_u+g_0|u|^{-2}+\frac{\lambda_u^3\rho^2}{1-\rho^2\lambda_u^2}\right)+o(\rho^{5/2}\Lambda).\end{aligned} \tag{7.22}$$

Let $u=\sqrt{\rho}k$. We have

$$\begin{aligned}&\lim_{\rho\to0}\sum_{u\in P_L}\left(\lambda_u+g_0|u|^{-2}+\frac{\lambda_u^3\rho^2}{1-\rho^2\lambda_u^2}\right)\rho^{-1/2}|\Lambda|^{-1}\\ &\quad=\lim_{\rho\to0}\frac{1}{(2\pi)^3}\int_{\varepsilon_L\le|k|\le\eta_L^{-1}} g_0|k|^{-2}\left(1-\frac{1}{\sqrt{1+4g_0|k|^{-2}}}\right)dk^3\\ &\quad=\pi^{-2}.\end{aligned} \tag{7.23}$$

So the leading term of right hand side of (7.21) is equal to $V_0g_0^{3/2}\pi^{-2}(\rho^{5/2}\Lambda)$. This completes the proof for (7.8). □

7.3 Proof of Lemma 4.4

Define $P(u,v)$ by

$$P(u,v)\equiv\sum_{\gamma\in M}f(\mathcal{A}^u\gamma)f(\mathcal{A}^v\gamma)\sqrt{(\gamma(u)+1)(\gamma(-u)+1)(\gamma(v)+1)(\gamma(-v)+1)}. \tag{7.24}$$

Recall $f(\alpha)=0$ when $|\alpha\rangle=0$ or $\alpha\notin M$.

Lemma 7.3 *Let $u,v\in\Lambda^*$, $u\neq v$ and $u,v\neq 0$.*

If one of u and $v\in P_L\cup P_I$, we have the following identity.

$$\langle\Psi|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\Psi\rangle=P(u,v). \tag{7.25}$$

If $u,v\in P_H$, we have

$$|\langle\Psi|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\Psi\rangle-P(u,v)|\le\text{const.}\,\rho^4\,|\lambda_u\lambda_v|\,. \tag{7.26}$$

Proof We first prove (7.25) and assume without loss of generality that $v\in P_L\cup P_I$. Using Lemma 7.1, we rewrite $\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle_\Psi$ as

$$\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle_\Psi=\sum_{\alpha\in M}f(\alpha)f(T(\alpha))\sqrt{(\alpha(u)+1)(\alpha(-u)+1)\alpha(v)\alpha(-v)}. \tag{7.27}$$

Here $|T(\alpha)\rangle=Ca_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle$ and C is positive normalization constant. Since $v\in P_L\cup P_I$ and $\alpha(v)>0,\alpha(-v)>0$, by definition of M there exists unique $\gamma\in M$ such that

$$\mathcal{A}^v\gamma=\alpha. \tag{7.28}$$

Therefore, with $|T(\alpha)\rangle=Ca_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle$, we have

$$T(\alpha)=\mathcal{A}^u\gamma. \tag{7.29}$$

Furthermore, by (7.28), we have

$$\gamma(u)=\alpha(u)\quad\text{and}\quad\gamma(v)=\alpha(v)+1. \tag{7.30}$$

Inserting (7.28), (7.29) and (7.30) into (7.27), we have proved (7.25).

To prove (7.26), we define N_v as the following set:

$$N_v\equiv\{\alpha\in M|\forall\gamma\in M,\ \mathcal{A}^v\gamma\neq\alpha\}\,. \tag{7.31}$$

Following the previous argument, we have

$$|\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle_\Psi-P(u,v)|\le\sum_{\alpha\in N_v,\beta\in N_u}|f(\alpha)f(\beta)\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle|. \tag{7.32}$$

The right hand side can be divided into two cases:

$$\sum_{\alpha\in N_v,\beta\in N_u,\beta(u)\beta(-u)\ge\alpha(v)\alpha(-v)}|f(\alpha)f(\beta)\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle|$$

$$+ \sum_{\alpha\in N_v,\beta\in N_u,\alpha(v)\alpha(-v)>\beta(u)\beta(-u)} |f(\alpha)f(\beta)\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle|. \tag{7.33}$$

By definition of f, if $\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle \neq 0$, we have $|f(\beta)| = |\lambda_u/\lambda_v f(\alpha)|$, $\beta(u) = \alpha(u)+1$ and $\beta(-u) = \alpha(-u)+1$. Denote by $N_{v,u} \subset N_v$ the set

$$N_{v,u} \equiv \{\alpha \in N_v : (\alpha(u)+1)(\alpha(-u)+1) \le \alpha(v)\alpha(-v)\}. \tag{7.34}$$

Hence we can bound (7.33) by

$$\left|\sum_{\alpha\in N_v,\beta\in N_v} f(\alpha)f(\beta)\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle\right|$$
$$\le \sum_{\alpha\in N_{v,u}} \left|\frac{\lambda_u}{\lambda_v}\right| |f(\alpha)|^2\alpha(v)\alpha(-v) + \sum_{\beta\in N_{u,v}} \left|\frac{\lambda_v}{\lambda_u}\right| |f(\beta)|^2\beta(u)\beta(-u). \tag{7.35}$$

Now we bound $\sum_{\alpha\in N_{v,u}} |f(\alpha)|^2\alpha(v)\alpha(-v)$. If $\alpha \in N_v$ and $\alpha(v)\alpha(-v) > 0$, then with Proposition 5.3, there exist α', $v' \in P_L$ with $\alpha' \in M^s_{v'}$ such that

$$\alpha = \mathcal{A}^{v',\, v-\frac{v'}{2}}\alpha'. \tag{7.36}$$

If $\alpha' \notin N_v$, then there exists γ' s.t. $\mathcal{A}^v\gamma' = \alpha'$. Hence

$$\mathcal{A}^v(\mathcal{A}^{v',\, v-\frac{v'}{2}}\gamma') = \alpha \quad \Rightarrow \quad \alpha \notin N_v$$

and we have a contradiction. Hence we have $\alpha' \in N_v$ and $\alpha'(-v) > 0$. Again by Proposition 5.3, there exist α'', $v'' \in P_L$ such that $\alpha'' \in M^s_{v''}$

$$\alpha' = \mathcal{A}^{v'',\, -v-\frac{v''}{2}}\alpha''. \tag{7.37}$$

Combining (7.36) and (7.37) and using (5.7), we express $f(\alpha)$ in terms of $f(\alpha'')$, $\alpha''(v')$, $\alpha''(v'')$ and $\alpha''(0)$ and λ's. By definition of M, $\alpha''(\widetilde{v}) \le m_c$ for any $\widetilde{v} \in P_L$ and we obtain

$$|f(\alpha)|^2 \le \text{const.}\, \rho^2 m_c^2 |\Lambda|^{-2}\lambda_v^2\, |\lambda_{-v+v'}\lambda_{v+v''}|\, f(\alpha'')^2. \tag{7.38}$$

By (5.11) and $-v+v'$, $v+v'' \in P_H$, we have

$$|f(\alpha)|^2 \le \text{const.}\, \rho^2 m_c^2 \lambda_v^2 \varepsilon_H^{-4} f(\alpha'')^2. \tag{7.39}$$

Summing over v', $v'' \in P_L$ and $\alpha'' \in M$, we obtain

$$\sum_{\alpha\in N_v,\alpha(v)+\alpha(-v)\ge 2} |f(\alpha)|^2 \le \text{const.}\, \rho^5\eta_L^{-6} m_c^2\lambda_v^2\varepsilon_H^{-4} \le (\rho^2\lambda_v)^2. \tag{7.40}$$

Similarly, one can prove that

$$\sum_{\alpha\in N_v,\alpha(v)+\alpha(-v)\ge m} |f(\alpha)|^2 \le (\rho^2\lambda_v)^m. \tag{7.41}$$

Hence, we can obtain

$$\sum_{\alpha\in N_{v,u}}\left|\frac{\lambda_u}{\lambda_v}\right||f(\alpha)|^2\alpha(v)\alpha(-v)\le\sum_{\alpha\in N_v}\left|\frac{\lambda_u}{\lambda_v}\right||f(\alpha)|^2\alpha(v)\alpha(-v)\le 2\rho^4\,|\lambda_u\lambda_v|\,.$$

Inserting this result into (7.35) and using the symmetry, we obtain

$$\left|\sum_{\alpha\in N_v,\beta\in N_v} f(\alpha)f(\beta)\langle\beta|a_u^\dagger a_{-u}^\dagger a_v a_{-v}|\alpha\rangle\right|\le\text{const.}\,\rho^4\,|\lambda_u\lambda_v|\,. \tag{7.42}$$

This completes the proof. □

Using this lemma, we can estimate the term $\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle$ as follows.

Lemma 7.4 *For $u, v\in P_I\cup P_H$,*

$$\left|\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle-\lambda_u\lambda_v\frac{Q_\Psi(0,0)-Q_\Psi(0)}{|\Lambda|^2}\right|$$
$$\le|\lambda_u\lambda_v|\,\rho^2((Q_\Psi(u,v)+Q_\Psi(u,-v))/2+Q_\Psi(u)+Q_\Psi(v)+\text{const.}\,\rho^2). \tag{7.43}$$

For $u\in P_L$, $v\in P_I\cup P_H$,

$$\left|\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle-\lambda_u\lambda_v\left(\frac{Q_\Psi(0,0)-Q_\Psi(0)}{|\Lambda|^2}+\frac{\rho^4\lambda_u^2}{1-\rho^2\lambda_u^2}\right)\right|$$
$$\le|\lambda_u\lambda_v|\,\rho^2\Bigg((Q_\Psi(u,v)+Q_\Psi(u,-v))/2+2Q_\Psi(v)$$
$$+\frac{4\rho^2\lambda_u^2}{1-\rho^2\lambda_u^2}(\rho^{\eta/2}+(\rho\lambda_u)^{2\sqrt{m_c}})\Bigg). \tag{7.44}$$

For $u, v\in P_L$

$$\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle-\lambda_u\lambda_v\frac{Q_\Psi(0,0)-Q_\Psi(0)}{|\Lambda|^2}$$
$$\le|\lambda_u\lambda_v|\,\rho^2\,(Q_\Psi(u,v)+2Q_\Psi(u)+2Q_\Psi(v)+3)\,. \tag{7.45}$$

We note that there is no absolute value on the left hand side of the inequality when $u, v\in P_L$.

Proof We first prove (7.43) concerning $u, v\in P_I\cup P_H$. By Lemma 7.3, we have

$$|\langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle-P(u,v)|\le\text{const.}\,\rho^4\,|\lambda_u\lambda_v|\,, \tag{7.46}$$

where $P(u,v)$ is defined in (7.24). By the property of f in (5.4), we can rewrite $P(u,v)$ as

$$\sum_{\gamma\in M,\mathcal{A}^u\gamma\in M,\mathcal{A}^v\gamma\in M}\lambda_u\lambda_v|f(\gamma)|^2\frac{\gamma(0)^2-\gamma(0)}{|\Lambda|^2}$$
$$\times\sqrt{(\gamma(u)+1)(\gamma(-u)+1)(\gamma(v)+1)(\gamma(-v)+1)}. \tag{7.47}$$

The situation that $\gamma \in M$ and $\mathcal{A}^{u(v)}\gamma \notin M$ can only happen when $\gamma(0) = 1$ or 0. But in this case, $\gamma(0)^2 - \gamma(0) = 0$ and the term vanishes. Hence the summation of γ in (7.47) can be replaced by $\sum_{\gamma \in M}$. Therefore, for $u, v \in P_I \cup P_H$, we have

$$\begin{aligned} &\left| P(u,v) - \lambda_u \lambda_v \frac{Q_\Psi(0,0) - Q_\Psi(0)}{|\Lambda|^2} \right| \\ &\quad \le \sum_{\gamma \in M} |\lambda_u \lambda_v| \, |f(\gamma)|^2 \frac{\gamma(0)^2}{|\Lambda|^2} \\ &\qquad \times \left| \sqrt{(\gamma(u)+1)(\gamma(-u)+1)(\gamma(v)+1)(\gamma(-v)+1)} - 1 \right|. \end{aligned} \tag{7.48}$$

From $\gamma(0) \le N$ and the Schwarz inequality, the rhs is bounded by

$$\sum_{\gamma \in M} |\lambda_u \lambda_v| \, |f(\gamma)|^2 \rho^2 \left[\left(\frac{\gamma(u) + \gamma(-u)}{2} + 1 \right) \left(\frac{\gamma(v) + \gamma(-v)}{2} + 1 \right) - 1 \right]. \tag{7.49}$$

By symmetry, we have $Q_\Psi(u) = Q_\Psi(-u)$ and $Q_\Psi(u,v) = Q_\Psi(-u,-v)$. So we have

$$(7.49) \le |\lambda_u \lambda_v| \, \rho^2 \left(\frac{1}{2} \left(Q_\Psi(u,v) + Q_\Psi(u,-v) \right) + Q_\Psi(u) + Q_\Psi(v) \right). \tag{7.50}$$

Together with (7.46), we have proved (7.43).

We now prove (7.44) concerning $u \in P_L$, $v \in P_I \cup P_H$. Following arguments in the previous paragraph and using (5.5) and (5.6), we can rewrite $P(u,v)$ as

$$\begin{aligned} &\sum_{\gamma \in M, \mathcal{A}^u \gamma \in M} \lambda_u \lambda_v |f(\gamma)|^2 \frac{\gamma(0)^2 - \gamma(0)}{|\Lambda|^2} \\ &\quad \times \sqrt{(\gamma^*(u)+1)(\gamma^*(-u)+1)(\gamma(v)+1)(\gamma(-v)+1)}. \end{aligned} \tag{7.51}$$

Notice that no matter we use (5.5) or (5.6), the final result is the same. For $\gamma \in M$ with $\gamma(0) \ge 2$, the case $\mathcal{A}^u \gamma \notin M$ can only happen when $\gamma^*(u) = m_c$. Hence, the summation of γ in (7.51) can be replaced by $\sum_{\gamma^*(u) \ne m_c}$. Since $\gamma^*(u) = \gamma^*(-u)$, for $u \in P_L$, $v \in P_I \cup P_H$ we have

$$P(u,v) = \sum_{\gamma^*(u) \ne m_c} \lambda_u \lambda_v |f(\gamma)|^2 \frac{\gamma(0)^2 - \gamma(0)}{|\Lambda|^2} (\gamma^*(u) + 1) \sqrt{(\gamma(v)+1)(\gamma(-v)+1)}. \tag{7.52}$$

Since $\gamma(0) \le N$, we have

$$\begin{aligned} &\left| P(u,v) - \lambda_u \lambda_v \frac{Q_\Psi(0,0) - Q_\Psi(0)}{|\Lambda|^2} - \sum_\gamma \lambda_u \lambda_v |f(\gamma)|^2 \frac{\gamma(0)^2 - \gamma(0)}{|\Lambda|^2} \gamma^*(u) \right| \\ &\quad \le \sum_{\gamma^*(u) \ne m_c} |\lambda_u \lambda_v| \, |f(\gamma)|^2 \rho^2 (\gamma^*(u) + 1) \left| \sqrt{(\gamma(v)+1)(\gamma(-v)+1)} - 1 \right| \\ &\qquad + \sum_{\gamma^*(u) = m_c} |\lambda_u \lambda_v| \, |f(\gamma)|^2 \rho^2 \left(\gamma^*(u) + 1 \right). \end{aligned} \tag{7.53}$$

We can replace $\sum_{\gamma^*(u)\neq m_c}$ in the first term of rhs by $\sum_{\gamma\in M}$ to have an upper bound. Since $\sqrt{(\gamma(v)+1)(\gamma(-v)+1)}-1\le[\gamma(v)+\gamma(-v)]/2$ and $\gamma^*(u)\le\gamma(u)+1$, we can bound the right hand side of (7.53) by

$$|\lambda_u\lambda_v|\,\rho^2\left[\frac{1}{2}\left(Q_\Psi(u,v)+Q_\Psi(u,-v)\right)+2Q_\Psi(v)+\sum_{\gamma(u)\ge m_c-1}2|f(\gamma)|^2\gamma(u)\right]. \tag{7.54}$$

The last term is bounded in (5.57), i.e.,

$$\sum_{\gamma(u)\ge m_c-1}|f(\gamma)|^2\gamma(u)\le\frac{\rho^2\lambda^2}{1-\rho^2\lambda^2}\,\rho^{\eta/2}. \tag{7.55}$$

The estimate (7.44) follows from last three inequalities and (7.51), provided that we can establish the following estimate

$$\sum_\gamma|f(\gamma)|^2\frac{\gamma(0)^2-\gamma(0)}{|\Lambda|^2}\gamma^*(u)=\frac{\rho^4\lambda_u^2}{(1-(\rho\lambda_u)^2)}\left[1+O(\rho^{\eta/2})+O((\rho\lambda_u)^{2\sqrt{m_c}})\right]. \tag{7.56}$$

To prove this, we first divide the summation of γ into $\gamma\in M_u^s$ and $\gamma\in M_u^a$. For the case $\gamma\in M_u^s$, we have

$$\sum_{\gamma\in M_u^s}|f(\gamma)|^2\frac{\gamma(0)^2-\gamma(0)}{|\Lambda|^2}\gamma^*(u)\le\rho^2Q_\Psi(u)\le\rho^2\frac{(\rho\lambda_u)^2}{(1-(\rho\lambda_u)^2)}(1+\rho^{2/3}), \tag{7.57}$$

where we have used (5.14) in the last inequality. For the case $\gamma\in M_u^a$, using (5.22), we have

$$\begin{aligned}\sum_{\gamma\in M_u^a}|f(\gamma)|^2\frac{\gamma(0)^2-\gamma(0)}{|\Lambda|^2}\gamma^*(u)&\le\text{const.}\,\rho^2\frac{\rho m_c}{\varepsilon_H}\sum_{\gamma\in M_u^s}|f(\gamma)|^2\gamma(u)\\&\le\rho^{\frac{8}{3}}\frac{(\rho\lambda_u)^2}{(1-(\rho\lambda_u)^2)}.\end{aligned} \tag{7.58}$$

This proves the upper bound part of (7.56). The lower bound follows from (5.58) since $\gamma^*(u)\ge\gamma(u)$.

Finally, we prove (7.45) concerning $u,v\in P_L$. Similar to the previous argument, by (5.5) and (5.6), we can rewrite $P(u,v)$ as

$$\begin{aligned}&\sum_{\gamma\in M,\mathcal{A}^u\gamma\in M,\mathcal{A}^v\gamma\in M}\lambda_u\lambda_v|f(\gamma)|^2\frac{\gamma(0)^2-\gamma(0)}{|\Lambda|^2}\\&\quad\times\sqrt{(\gamma^*(u)+1)(\gamma^*(-u)+1)(\gamma^*(v)+1)(\gamma^*(-v)+1)}.\end{aligned} \tag{7.59}$$

Since $\lambda_u\lambda_v\ge0$ and $\gamma^*(u)=\gamma^*(-u)$, we have for $u,v\in P_L$,

$$P(u,v)-\lambda_u\lambda_v\frac{Q_\Psi(0,0)-Q_\Psi(0)}{|\Lambda|^2}\le\sum_{\gamma\in M}\lambda_u\lambda_v\rho^2\left|(\gamma^*(u)+1)(\gamma^*(v)+1)-1\right|. \tag{7.60}$$

Using $\gamma^*-\gamma\le1$, we have proved (7.45). □

We now can now prove Lemma 4.4.

Proof Summing over $u, v \neq 0$ of (7.43), (7.44) and (7.45), we obtain that

$$\sum_{u,\, v\neq 0} \frac{V_{u-v}}{|\Lambda|^2} \langle a_u^\dagger a_{-u}^\dagger a_v a_{-v}\rangle \leq A + B + \Omega \tag{7.61}$$

where

$$\begin{aligned}
A &= \frac{Q_\Psi(0,0) - Q_\Psi(0)}{|\Lambda|^2} \sum_{u,v\neq 0} \frac{V_{u-v}}{|\Lambda|^2} \lambda_u \lambda_v, \\
B &= 2 \sum_{u\in P_L, v\in P_I\cup P_H} \frac{V_{u-v}}{|\Lambda|^2} \lambda_u \lambda_v \frac{\rho^4 \lambda_u^2}{1-\rho^2\lambda^2}, \\
\Omega &= \frac{1}{|\Lambda|^2}\Bigg(\sum_{u,\, v\neq 0} |V_{u-v}||\lambda_u\lambda_v|\rho^2 Q_\Psi(u,v) + \sum_{u\in P_I\cup P_H, v\neq 0} 4|\lambda_u \lambda_v V_{u-v}|\rho^2 Q_\Psi(u) \\
&\quad + \sum_{u,\, v\in P_L} 3|V_{u-v}||\lambda_u\lambda_v|\rho^2 (Q_\Psi(u)+1) + \sum_{u,v\in P_I\cup P_H} \text{const.}\, \rho^4 |\lambda_u\lambda_v||V_{u-v}| \\
&\quad + \sum_{u\in P_L, v\in P_I\cup P_H} |\lambda_u\lambda_v||V_{u-v}| \frac{4\rho^4\lambda_u^2}{1-\rho^2\lambda_u^2} (\rho^{\eta/2} + (\lambda_u\rho)^{2\sqrt{m_c}})\Bigg).
\end{aligned} \tag{7.62}$$

The error term Ω can be bounded by using the following facts, (1) $|\rho\lambda_u| \leq 1$, (2) $|\sum_{v\neq 0} \lambda_v V_{u-v}| \leq \text{const.}\, \Lambda$, (3) $|V_u| \leq V_0$, (4) $|\lambda_u| \leq g_0 |u|^{-2}$ for any $u \neq 0$ and (5) $\sum_{u,v} |\lambda_u V_{u-v} \lambda_v| \leq \text{const.}\, |\Lambda|^2$:

$$\begin{aligned}
\Omega \leq \frac{\text{const.}}{|\Lambda|^2}\Bigg(&\sum_{u,\, v\neq 0} Q_\Psi(u,v) + \sum_{u\in P_I\cup P_H} Q_\Psi(u)\rho\Lambda + \sum_{u,\, v\in P_L} \frac{Q_\Psi(u)+1}{u^2 v^2}\rho^2 \\
&+ \rho^4 |\Lambda|^2 + \sum_{u\in P_L} \frac{\rho^3\lambda_u^2}{1-\rho^2\lambda_u^2} \Lambda (\rho^{\eta/2} + (\lambda_u\rho)^{2\sqrt{m_c}})\Bigg).
\end{aligned} \tag{7.63}$$

By (6.1) and (5.2), the first two terms on the right hand side are bounded by $o(\rho^{5/2})$. Using the trivial bound $Q_\Psi(u) \leq m_c$ for $u \in P_L$, the third term is also bounded by $o(\rho^{5/2})$. By (7.19) and (7.20), the last term is also $o(\rho^{5/2})$. Hence the error terms are bounded by $\Omega \leq o(\rho^{5/2})$.

We now estimate A and B. Notice that $(Q_\Psi(0,0) - Q_\Psi(0))|\Lambda|^{-2} = \rho_0^2 + o(\rho^{5/2})$. Hence we shall replace this factor in A by ρ_0^2. Since $\lambda_u = -w_u$ for $u \in P_I \cup P_H$, we have

$$\sum_{u,v\neq 0} \lambda_u\lambda_v = \sum_{u,v\neq 0} w_u w_v - 2 \sum_{u\in P_L, v\neq 0} (\lambda_u + w_u) w_v + \sum_{u,v\in P_L} (\lambda_u + w_u)(\lambda_v + w_v).$$

We can now decompose A into

$$A = \|w^2 V\|_1 \rho_0^2 + A_1 + A_2 + A_3 + o(\rho^{5/2}) \tag{7.64}$$

where

$$A_1 = -2\rho_0^2 \sum_{u\in P_L, v\in P_I\cup P_H} \frac{V_{u-v}}{|\Lambda|^2}(\lambda_u + w_u)w_v,$$

$$A_2 = -2\rho_0^2 \sum_{u\in P_L, v\in P_L} \frac{V_{u-v}}{|\Lambda|^2}(\lambda_u + w_u)w_v,$$

$$A_3 = \sum_{u,v\in P_L} \frac{V_{u-v}}{|\Lambda|^2}(\lambda_u + w_u)(\lambda_v + w_v)\rho_0^2.$$

Since $|w_u\rho| \le \text{const.}\, \rho|u|^{-2} \le \varepsilon_L^{-2}$, we have $A_3 \le o(\rho^{5/2})$. We can also obtain the simple estimate $A_2 \le o(\rho^{5/2})$.

If we replace ρ^2 in B by ρ_0^2, which is equal to $\rho^2 - O(\rho^{5/2})$, we have

$$B + A_1 = -2 \sum_{u\in P_L, v\in P_I\cup P_H} \frac{V_{u-v}}{|\Lambda|^2} w_v \rho^2 \left(\lambda_u \frac{\rho^2\lambda_u^2}{1-\rho^2\lambda_u^2} + \lambda_u + w_u\right). \tag{7.65}$$

Using $|V_{u-v} - V_v| \le \text{const.}\, |u|$ for $u \in P_L$ and $v \in P_I \cup P_H$, we can simplify $B + A_1$ as

$$B + A_1 \le -\frac{2\|Vw\|_1}{|\Lambda|}\rho^2 \sum_{u\in P_L}\left(\frac{\lambda_u}{1-\rho^2\lambda_u^2} + w_u\right) + o(\rho^{5/2}). \tag{7.66}$$

Since $|g_u - g_0| \le \text{const.}\, |u|$, we have $|w_u - g_0|u|^{-2}| \le \text{const.}\, \rho^{-1/2}\varepsilon_L^{-1}$. Then we can replace w_u with $g_0|u|^{-2}$ in (7.66). Setting $u = \rho^{1/2}k$, we have, by definition of λ,

$$\begin{aligned} &\lim_{\rho\to 0}(\rho^{1/2}\Lambda)^{-1} \sum_{u\in P_L}\left(\frac{\lambda_u}{1-\rho^2\lambda_u^2} + g_0|u|^{-2}\right) \\ &\quad = \frac{1}{8\pi^3}\int_{k\in\mathbb{R}^3} g_0|k|^{-2}\left(\frac{\sqrt{1+4g_0|k|^{-2}}-1}{\sqrt{1+4g_0|k|^{-2}}}\right) dk^3 = \frac{g_0^{3/2}}{\pi^2}. \end{aligned} \tag{7.67}$$

Inserting this result into (7.66) and (7.64), we have proved (4.11). □

8 Proof of Lemma 4.5

In this section, we prove Lemma 4.5 concerning potential energy terms with one a_0. Let $v_j \in \Lambda^*$ and $v_j \ne 0$ for $j = 1, 2, 3$. Define $P_{H,c}$ as the following subset of P_H:

$$P_{H,c} = \{k \in P_H : |k| \le k_c\}. \tag{8.1}$$

The following lemma classify all possible scenarios of v_1, v_2, v_3. Through out this section, we assume that $v_i \ne 0$ for $i = 1, 2, 3$.

Lemma 8.1 *Suppose $\beta, \alpha \in M$ and $\langle\alpha|a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}|\beta\rangle \ne 0$. Then there are only three possibilities*:

1.

$$v_1 \in P_L, \quad v_2, v_3 \in P_{H,c}, \quad v_i \ne \pm v_j \quad \textit{for } i \ne j. \tag{8.2}$$

2.

$$v_1 \in P_{H,c}, \quad v_2 \in P_L, \quad v_3 \in P_{H,c}, \quad v_i \neq \pm v_j \quad \textit{for } i \neq j; \quad \text{or} \quad 2 \leftrightarrow 3. \tag{8.3}$$

3.

$$v_1 \in P_L, \quad v_2 \in P_L, \quad v_3 \in P_L. \tag{8.4}$$

Proof Since particles with momenta in P_I are always created in pair, e.g., $(u, -u)$, either none of v_i's belongs to P_I or two of them belong to P_I. Thus we have:

$$v_1, v_2 \in P_I \quad \Rightarrow \quad v_1 = v_2, \quad \text{or} \quad 2 \leftrightarrow 3, \tag{8.5}$$

$$v_2, v_3 \in P_I \quad \Rightarrow \quad v_2 = -v_3. \tag{8.6}$$

If two of v_i's are in P_I, by the momentum conservation $v_1 = v_2 + v_3$ the other one must be equal to zero, which is a contradiction. Therefore

$$v_i \notin P_I, \quad \text{for } 1 \le i \le 3. \tag{8.7}$$

The restriction $|v_i| \le k_c$ follows from the construction of M. Therefore, we have

$$v_i \in P_L \cup P_{H,c}, \quad \text{for } 1 \le i \le 3. \tag{8.8}$$

Since particles in $P_{H,c}$ are always created in soft pair creations which generated two particles in $P_{H,c}$, the number of particles in $P_{H,c}$ is even. So either none of v_i's are in $P_{H,c}$ or two of them are in $P_{H,c}$. Together with (8.8), and momentum conservation, we prove the lemma. □

For fixed v_1, v_2, v_3, define

$$F(\alpha) \equiv \sum_{i: v_i \in P_L, i=1,2,3} |\alpha(v_i) - \alpha(-v_i)|. \tag{8.9}$$

Lemma 8.2 *For any $\alpha, \beta \in M$ if $\langle \alpha | a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3} | \beta \rangle \neq 0$ and $v_i \neq \pm v_j$, we have*:

$$F(\alpha) + F(\beta) = \#\{i = 1, 2, 3 : v_i \in P_L\}. \tag{8.10}$$

Furthermore, the ratio between $f(\alpha)$ and $f(\beta)$ is bounded as follows.

$$\rho^{\frac{1}{20}} \sqrt{N}^{F(\alpha)-F(\beta)} \le \left| \frac{f(\beta)\sqrt{\lambda_{v_1}}}{f(\alpha)\sqrt{\lambda_{v_2}\lambda_{v_3}\alpha(0)/\Lambda}} \right| \le \rho^{\frac{-1}{20}} \sqrt{N}^{F(\alpha)-F(\beta)}. \tag{8.11}$$

Proof Since $v_i \neq \pm v_j$, for each i fixed, if $\alpha \in M_{v_i}^a$, then $\beta \in M_{v_i}^s$ and vice verse. This proves (8.10).

Recall the definition of f in (3.18). Then one can check the ratio involving $f(\beta)/f(\alpha)$ in (8.11) depends only on the last factor

$$\prod_{u \in P_L, \alpha^*(u) - \alpha(u) = 1} \sqrt{4\alpha^*(u)\lambda_u |\Lambda|^{-1}}.$$

We now use (5.10) to bound λ in this expression. Since $F(\alpha)$ counts how many times this factor appears, this proves (8.11). □

Using the definitions of η_L and m_c, the bound $\alpha(0)/\Lambda \le \rho$ and lemma 7.1, we have

$$|f(\alpha)f(\beta)\langle\alpha|a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}|\beta\rangle| \le \sqrt{N}^{F(\alpha)-F(\beta)+1}\rho^{\frac{-1}{20}}\sqrt{\rho\left|\frac{\lambda_{v_2}\lambda_{v_3}}{\lambda_{v_1}}\right|}\sqrt{\alpha(v_1)(\alpha(v_2)+1)(\alpha(v_3)+1)}|f(\alpha)|^2 \quad (8.12)$$

and

$$|f(\alpha)f(\beta)\langle\alpha|a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}|\beta\rangle| \le \sqrt{N}^{F(\beta)-F(\alpha)+1}\rho^{\frac{-1}{20}}\sqrt{|\lambda_{v_1}\lambda_{v_2}^{-1}\lambda_{v_3}^{-1}|\rho^{-1}}\sqrt{(\beta(v_1)+1)\beta(v_2)\beta(v_3)}|f(\beta)|^2. \quad (8.13)$$

Lemma 4.5 follows from summing the three inequalities of the following lemma.

Lemma 8.3 *In the limit $k_c \to \infty$, $\rho \to 0$, we have*

$$\overline{\lim_{k_c,\,\rho}}\,|\Lambda|^{-2}\rho^{-5/2}\sum_{(8.2)}\langle V_{v_2}a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}\rangle = -2\|Vw\|_1\frac{g_0^{3/2}}{3\pi^2}, \quad (8.14)$$

$$\overline{\lim_{k_c,\,\rho}}\,|\Lambda|^{-2}\rho^{-5/2}\sum_{(8.3)}|V_{v_2}\langle a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}\rangle| = 0, \quad (8.15)$$

$$\overline{\lim_{k_c,\,\rho}}\,|\Lambda|^{-2}\rho^{-5/2}\sum_{(8.4)}|V_{v_2}\langle a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}\rangle| = 0. \quad (8.16)$$

Proof We first prove (8.14) concerning (8.2), which implies that $F(\alpha)+F(\beta)=1$. By the bounds on λ_u in (5.10) and $\alpha^*(u) \le m_c$ for $u \in P_L$, we have, for $F(\beta)=0$ the following slightly modified version of (8.13)

$$|f(\alpha)f(\beta)||\langle\alpha|a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}|\beta\rangle| \le \rho^{\frac{-1}{10}}\rho^{-1}\sqrt{|\lambda_{v_2}^{-1}\lambda_{v_3}^{-1}|}\sqrt{\beta(v_2)\beta(v_3)}|f(\beta)|^2. \quad (8.17)$$

Here we replaced $\rho^{-1/20}$ in (8.13) by $\rho^{-1/10}$ to accommodate small errors. Summing over β with $F(\beta)=0$, we have

$$\sum_{F(\beta)=0} f(\beta)f(\alpha)|\langle\alpha|a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}|\beta\rangle| \le \rho^{-11/10}\sqrt{|\lambda_{v_2}^{-1}\lambda_{v_3}^{-1}|}Q_\Psi(u,v). \quad (8.18)$$

Using the bound (5.42) on $Q_\Psi(u,v)$ and $|\lambda_u| \le g_0|u|^{-2}$, we obtain that (8.18) $= o(\rho^2)$.

Since $F(\alpha)+F(\beta)=1$, the other case is $F(\alpha)=0$. Hence we have

$$\langle a_0^\dagger a_{v_1}^\dagger a_{v_2}a_{v_3}\rangle = A_1 + A_2 + o(\rho^{3/2}), \quad (8.19)$$

$$A_1 = \sum_{F(\alpha)=0}\sqrt{\alpha(0)\alpha(v_1)}f(\alpha)f(\beta),$$

$$A_2 = \sum_{F(\alpha)=0}\sqrt{\alpha(0)\alpha(v_1)}\Big(\sqrt{(\alpha(v_2)+1)(\alpha(v_3)+1)}-1\Big)f(\alpha)f(\beta).$$

By the estimate (8.11) and the Schwarz inequality $|2(\sqrt{(a+1)(b+1)}-1)| \le a+b$, we have

$$|A_2| \le \rho^{4/5} \sum_{F(\alpha=0)} \frac{\alpha(v_2)+\alpha(v_3)}{2} |f(\alpha)|^2$$
$$\le \rho^{4/5}(Q_\Psi(v_2)+Q_\Psi(v_3)) \le o(\rho^2), \tag{8.20}$$

where we have used the bounds on λ's and $Q_\Psi(u)$ for $u \in P_H$.

By the property (5.7) for f, we have

$$A_1 = 2\sqrt{\lambda_{v_2}\lambda_{v_3}} \sum_{F(\alpha)=0} \alpha(0)\alpha(v_1)|\Lambda|^{-1}|f(\alpha)|^2. \tag{8.21}$$

We notice

$$\sum_{F(\alpha)=0} \alpha(0)\alpha(v_1)|f(\alpha)|^2 = Q_\Psi(0,v_1)|\Lambda|^{-1} - \sum_{\alpha\in M_{v_1}^a} \alpha(0)\alpha(v_1)|\Lambda|^{-1}|f(\alpha)|^2. \tag{8.22}$$

The absolute value of the second term is less than $\rho m_c \sum_{\alpha\in M_{v_1}^a} |f(\alpha)|^2$. By (5.23), it is less than $\rho^{7/4}$. Then with $|\sqrt{\lambda_{v_2}\lambda_{v_3}}| \le O(\varepsilon_H^{-2})$, we obtain

$$\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle = 2\sqrt{\lambda_{v_2}\lambda_{v_3}}\, Q_\Psi(0,v_1)|\Lambda|^{-1} + o(\rho^{3/2}). \tag{8.23}$$

Recall $\lambda_u = -w_u$ for $u \in P_I \cup P_H$ and $w_u = w_{-u}$ due to our assumption on V. Since $v_1 \le P_L \sim \sqrt{\rho}$ and $v_2 = -v_3 + v_1$ and $v_2 \in P_{H,c}$, we can check that

$$\left|\lambda_{v_2} - \lambda_{v_3}\right| \le \rho^{1/3}. \tag{8.24}$$

Inserting this in (8.23), we arrive at

$$\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle = 2\lambda_{v_2} Q_\Psi(0,v_1)|\Lambda|^{-1} + o(\rho^{5/4}). \tag{8.25}$$

In the limit $k_c \to \infty$, $\rho \to 0$, we have

$$|\Lambda|^{-2} \sum_{v_1\in P_L, v_2\in P_{H,c}} \langle V_{v_2} a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle = -\|Vw\|_1 |\Lambda|^{-2} \sum_{v_1\in P_L} Q_\Psi(0,v_1) + o(\rho^{5/2}). \tag{8.26}$$

We note

$$|\Lambda|^{-2}\sum_{v_1\in P_L} Q_\Psi(0,v_1) = \rho|\Lambda|^{-1}Q_\Psi(0) - |\Lambda|^{-2}Q_\Psi(0,0) - |\Lambda|^{-2}\sum_{u\in P_I\cup P_H} Q_\Psi(0,u). \tag{8.27}$$

The last term is less than $N|\Lambda|^{-2}\sum_{u\in P_I\cup P_H} Q_\Psi(u) \le o(\rho^{5/2})$ by Theorem 5.1. Together with Lemmas 5.6, 5.3 on $Q_\Psi(0,0)$ and $Q_\Psi(0)$, we can compute the first two terms, i.e.,

$$|\Lambda|^{-2}\sum_{v_1\in P_L} Q_\Psi(0,v_1) = \rho_0(\rho-\rho_0) + o(\rho^{5/2}). \tag{8.28}$$

This yields (8.14).

We next prove (8.15) concerning (8.3). Without loss of generality we assume that

$$v_{1,3} \in P_{H,c} \quad \text{and} \quad v_2 \in P_L. \tag{8.29}$$

Following similar arguments in the previous proof, i.e., using Lemma 7.1, (8.12) or (8.13) and the bounds on λ_u's, we have

$$\begin{aligned} |\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle| \le & \sum_{F(\alpha)=0} \rho^{-\frac{1}{10}} \sqrt{\alpha(v_1)(\alpha(v_3)+1)|\lambda_{v_1}^{-1}|}|f(\alpha)|^2 \\ & + \sum_{F(\beta)=0} \rho^{-\frac{1}{10}} \sqrt{\beta(v_3)(\beta(v_1)+1)|\lambda_{v_3}^{-1}|}|f(\beta)|^2. \end{aligned} \tag{8.30}$$

For the upper bound, we can replace $\sum_{F(\alpha)=0}$ by $\sum_{\alpha\in M}$. Using the upper bounds (5.15) and (5.42) on $Q_\Psi(u)$ and $Q_\Psi(u,v)$ for $u,v \in P_H$, we obtain $|\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle| \le \text{const.}\, \rho^{3/2}$. This proves (8.15).

We now prove (8.16) concerning $v_i \in P_L$ satisfying $F(\alpha)+F(\beta)=3$. It is easy to prove that the contribution from the special cases, $v_1 = -v_2$ (or v_3) or $v_2 = v_3$, is negligible,

$$\overline{\lim_{\rho}} \sum_{\text{special cases}} |V_{v_2}\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle| \rho^{-5/2}|\Lambda|^{-2} = 0. \tag{8.31}$$

So from now on we assume that $v_i \ne \pm v_j$ for $i \ne j$. As before, we rewrite $\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle$ by using Lemma 7.1 and (8.12) or (8.13). Together with the bounds on λ_u's and $\alpha(v_i) \le m_c$, we have

$$\begin{aligned} |\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle| \le & \sum_{F(\alpha)=0} N^{-1}\rho^{-\frac{1}{10}}|f(\alpha)|^2 + \sum_{F(\alpha)=1} \rho^{-\frac{1}{10}}|f(\alpha)|^2 \\ & + \sum_{F(\beta)=0} N^{-1}\rho^{-\frac{1}{10}}|f(\beta)|^2 + \sum_{F(\beta)=1} \rho^{-\frac{1}{10}}|f(\beta)|^2. \end{aligned} \tag{8.32}$$

By symmetry, we only need to estimate the first two terms on the rhs The first term is less than $N^{-1}\rho^{-\frac{1}{10}}$. For the second term, we note $F(\alpha)=1$ implies that there exists i, $1 \le i \le 3$ such that $\alpha \in M_{v_i}^a$. By (5.23), we have

$$\sum_{F(\alpha)=1} |f(\alpha)|^2 \le \rho^{3/4}. \tag{8.33}$$

This implies $|\langle a_0^\dagger a_{v_1}^\dagger a_{v_2} a_{v_3}\rangle| \le \rho^{1/2}$ and (8.16), which complete the proof. □

9 Interaction Energy with Four Nonzero Momenta: The Classification

In the next three sections, we will prove Lemma 4.6 involving interaction energy without a_0. We will show that the only contribution to the accuracy we need comes from four high momentum particles, to be computed in next section. In this section, we start the procedure of identifying the error terms.

For $\alpha, \beta \in M$, we have the following lemma, similar to Lemma 8.1 and Lemma 8.2. Since it can be proved by same method, we will only state the result.

Lemma 9.1 *Suppose* $v_i \neq 0, 1 \leq i \leq 4$ *and* $v_1 + v_2 \neq 0$, $v_1 \neq v_3$ *or* v_4. *If* $\langle\alpha|a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}|\beta\rangle \neq 0$ *for some* $\alpha, \beta \in M$, *then there are exactly four cases*:

1. *All of* $v_i \in P_L$ *for* $1 \leq i \leq 4$.
2. $v_1, v_2 \in P_L$, $v_3, v_4 \in P_{H,c}$.
3. *One of* v_1, v_2 *is in* P_L *and the other is in* $P_{H,c}$; *one of* v_3, v_4 *is in* P_L *and the other is in* $P_{H,c}$.
4. *All of* $v_i \in P_{H,c}$ *for* $1 \leq i \leq 4$.

If $v_i \neq \pm v_j$, *for* $1 \leq i, j \leq 4$, *we have*

$$\rho^{\frac{1}{20}} \sqrt{N}^{F(\alpha)-F(\beta)} \leq \left| \frac{f(\beta)\sqrt{\lambda_{v_1}\lambda_{v_2}}}{f(\alpha)\sqrt{\lambda_{v_3}\lambda_{v_4}}} \right| \leq \rho^{\frac{-1}{20}} \sqrt{N}^{F(\alpha)-F(\beta)}, \tag{9.1}$$

$$\begin{aligned} &|f(\alpha)f(\beta)\langle\alpha|a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}|\beta\rangle| \\ &\quad\leq \sqrt{N}^{F(\alpha)-F(\beta)} \rho^{\frac{-1}{20}} \sqrt{\frac{\lambda_{v_3}\lambda_{v_4}}{\lambda_{v_1}\lambda_{v_2}}} \sqrt{\alpha(v_1)\alpha(v_2)(\alpha(v_3)+1)(\alpha(v_4)+1)}|f(\alpha)|^2 \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} &|f(\alpha)f(\beta)\langle\alpha|a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}|\beta\rangle| \\ &\quad\leq \sqrt{N}^{F(\beta)-F(\alpha)} \rho^{\frac{-1}{20}} \sqrt{\frac{\lambda_{v_1}\lambda_{v_2}}{\lambda_{v_3}\lambda_{v_4}}} \sqrt{(\beta(v_1)+1)(\beta(v_2)+1)\beta(v_3)\beta(v_4)}|f(\beta)|^2. \end{aligned} \tag{9.3}$$

Proposition 9.1 *For* $u \in P_L$ *and* $v \in P_{H,c}$, *we have the following inequality*

$$\sum_{\alpha \in M^a_u} \alpha(v)|f(\alpha)^2| \leq |\lambda_v|\rho^{3-\frac{1}{10}}. \tag{9.4}$$

Proof By definition of M (3.9), for any $\alpha \in M^a_u$, there exist $\beta \in M^s_u$ and k such that $\mathcal{A}^{u,k}\beta = \alpha$ and $\pm k + u/2 \in P_{H,c}$. Clearly, for any $v \in P_H$ we have $\alpha(v) \leq \beta(v) + 1$ and the case we need the constant 1 occurs only when $v = k + u/2$ or $v = -k + u/2$. Hence we can bound the left hand side of (9.4) by

$$\sum_\beta \sum_{k:\pm k+u/2 \in P_{H,c}} \beta(v)|f(\mathcal{A}^{u,k}\beta)^2| + \sum_\beta \sum_{k:\pm k+u/2=v} |f(\mathcal{A}^{u,k}\beta)^2|. \tag{9.5}$$

Recall (5.7) implies that

$$|f(\mathcal{A}^{u,k}\beta)|^2 \leq |f(\beta)|^2 \rho m_c |\Lambda|^{-1} |\lambda_{k+u/2}\lambda_{-k+u/2}|. \tag{9.6}$$

By the bound (5.11) on λ, we obtain that

$$\begin{aligned} (9.5) &\leq \sum_\beta \beta(v)|f(\beta)|^2 \frac{\rho m_c}{|\Lambda|} \left[\sum_{\pm k+u/2 \in P_H} |\lambda_{k+u/2}\lambda_{-k+u/2}| \right] + |\lambda_v||\Lambda|^{-1} \\ &\leq Q_\Psi(v)\rho m_c \varepsilon_H^{-4} k_c^3 + |\lambda_v||\Lambda|^{-1}. \end{aligned} \tag{9.7}$$

Using Proposition 5.4, we have proved (9.4). □

Lemma 9.2 *We have the following estimates on the interaction energies*:

$$\overline{\lim_{m_c,\,\rho}}\, \rho^{-5/2}|\Lambda|^{-2} \sum_{v_1,v_2,v_3,v_4\in P_L} |V_{v_1-v_3}\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| = 0, \tag{9.8}$$

$$\overline{\lim_{m_c,\,\rho}}\, \rho^{-5/2}|\Lambda|^{-2} \sum_{v_1+v_2\neq 0,\, v_1,v_2\in P_L,\, v_3,v_4\in P_H} |V_{v_1-v_3}\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| = 0, \tag{9.9}$$

$$\overline{\lim_{m_c,\,\rho}}\, \rho^{-5/2}|\Lambda|^{-2} \sum_{v_1,v_3\in P_L,\, v_2,v_4\in P_H} |V_{v_1-v_3}\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| = 0. \tag{9.10}$$

In other words, *the contributions from case* 1, 2 *and* 3 *in Lemma* 9.1 *are negligible for our purpose*.

Proof We first prove the (9.8) concerning $v_i \in P_L$. By Lemma 7.1, we have

$$|\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| \le \sum_\alpha |f(\alpha)f(T(\alpha))|m_c^4. \tag{9.11}$$

Using the Schwarz inequality, we have $|\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| \le m_c^4$. The summation over the v_i with $v_i = \pm v_j$ for some $1 \le i < j \le 4$ is negligible in the sense that

$$|\Lambda|^{-2} \sum_{v_1,v_2,v_3,v_4\in P_L,\, v_i=\pm v_j} |\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| \le o(\rho^{5/2}). \tag{9.12}$$

From now on, we assume that $v_i \neq \pm v_j$ for any $1 \le i < j \le 4$.

Using (9.2), (9.3) and the bounds (5.10) on λ, we have

$$\begin{aligned}|\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| \le{}& \sum_{F(\alpha)\le 1} \rho^{\frac{-1}{10}} N^{-1}|f(\alpha)^2| \\ &+ \sum_{F(\alpha)=2} \rho^{\frac{-1}{10}}|f(\alpha)^2| + \sum_{F(\beta)\le 1} \rho^{\frac{-1}{10}} N^{-1}|f(\beta)^2|.\end{aligned}$$

By (5.24), we have $|\langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi| \le \rho^{9/5}$. Together with (9.12) and $\Lambda = \rho^{-25/8}$, we can sum over v_j to have

$$|\Lambda|^{-2} \sum_{v_1,v_2,v_3,v_4\in P_L} \langle a^\dagger_{v_1}a^\dagger_{v_2}a_{v_3}a_{v_4}\rangle_\Psi \le o(\rho^{5/2}). \tag{9.13}$$

We now prove (9.9) concerning $v_{1,2} \in P_L$ and $v_{3,4} \in P_{H_c}$. As before, by (9.2), (9.3), (5.10) and (5.11), we have

$$\begin{aligned}|\langle a^\dagger_{u_1}a^\dagger_{u_2}a_{u_3}a_{u_4}\rangle_\Psi| ={}& \sum_{F(\alpha)=0} N^{-1}\rho^{\frac{9}{10}}\sqrt{(\alpha(v_3)+1)(\alpha(v_4)+1)}|f(\alpha)|^2 \\ &+ \sum_{F(\beta)\le 1} \rho^{\frac{-11}{10}}\sqrt{\frac{\beta(v_3)\beta(v_4)}{\lambda_{v_3}\lambda_{v_4}}}|f(\beta)|^2.\end{aligned}$$

By the Schwarz inequality, we have that the first term in rhs is $o(\rho^4)$. Since $v_3, v_4 \in P_H$, by (5.42) we obtain that the second term in rhs is $o(\rho^{11/4})$. So

$$|\langle a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}\rangle_\Psi| \le \rho^{\frac{11}{4}}. \tag{9.14}$$

Summing over v_j's, we have proved (9.8).

Finally, we prove (9.10) concerning $v_{1,3} \in P_L$ and $v_{2,4} \in P_H$. Again, with (9.2), (9.3) and the bounds on λ's in (5.10) and (5.11), we have

$$|\langle a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}\rangle_\Psi| \le Q_1 + Q_2 + Q_3, \tag{9.15}$$

$$Q_1 = \sum_{F(\alpha)=0} N^{-1}\rho^{\frac{-1}{10}} \sqrt{\frac{\alpha(v_2)(\alpha(v_4)+1)}{|\lambda_{v_2}|}}|f(\alpha)|^2, \tag{9.16}$$

$$Q_2 = \sum_{F(\beta)=0} N^{-1}\rho^{\frac{-1}{10}} \sqrt{\frac{\alpha(v_4)(\alpha(v_2)+1)}{|\lambda_{v_4}|}}|f(\beta)|^2, \tag{9.17}$$

$$Q_3 = \sum_{F(\alpha)=1} \rho^{\frac{-1}{10}} \sqrt{\frac{\alpha(v_2)(\alpha(v_4)+1)}{|\lambda_{v_2}|}}|f(\alpha)|^2. \tag{9.18}$$

By Theorem 5.1 and the fact $\sqrt{x} \le x$ for $x \in \mathbb{N}$, we have

$$Q_1 \le N^{-1}\rho^{\frac{-1}{10}}\lambda_{v_2}^{-1/2}(Q_\Psi(v_2) + Q_\Psi(v_2, v_4)) \le \rho^3,$$

where we have used the bounds (5.15) and (5.42) on $Q_\Psi(u)$ and $Q_\Psi(u, v)$. Similarly, we have $Q_2 \le \rho^3$. Again using the fact $\sqrt{x} \le x$ for $x \in \mathbb{N}$, we have

$$\begin{aligned} Q_3 &\le \sum_{F(\alpha)=1} \rho^{\frac{-1}{10}}\alpha(v_2)|\lambda_{v_2}|^{-1/2}|f(\alpha)|^2 + \rho^{-\frac{1}{10}}|\lambda_{v_2}|^{-1/2}Q_\Psi(v_2, v_4) \\ &\le \sum_{F(\alpha)=1} \rho^{\frac{-1}{10}}\alpha(v_2)|\lambda_{v_2}|^{-1/2}|f(\alpha)|^2 + \rho^3, \end{aligned}$$

where we have used (5.42). We can estimate the first term in rhs by (9.4). Collecting all these bounds, we have proved that

$$|\langle a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}\rangle_\Psi| \le \rho^{2.7}. \tag{9.19}$$

Summing over v_j's, we have proved (9.10). □

10 Interaction Energy with Four High Momentum Legs I: The Main Term

We now estimate of the interaction energy in the case 4 of Lemma 9.1, i.e., $k_i, i = 1, 2, 3, 4$ satisfy

$$k_1 + k_2 = k_3 + k_4, \quad k_1 + k_2 \ne 0, \quad k_1 \ne k_3, \quad k_1 \ne k_4, \quad k_i \in P_{H,c}. \tag{10.1}$$

In the remainder of this paper, all p_i's, q_i's, k_i's belong to $P_{H,c}$ and u_i, v_i's belong to P_L. We start with some special cases.

Lemma 10.1 *Suppose k_i satisfy* (10.1). *Then we have*

$$\sum_{k_1,k_3} |V_{k_1-k_3}\langle a_{k_1}^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4}\rangle| = o(\rho^{5/2}|\Lambda|^2), \tag{10.2}$$

$$\sum_{k_1,k_2} |V_{2k_1}\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{-k_1} a_{k_4}\rangle| = o(\rho^{5/2}|\Lambda|^2). \tag{10.3}$$

Proof By definition of f, if $\langle\alpha|a_{k_1}^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4}|\beta\rangle \neq 0$, then

$$f(\alpha) = \sqrt{\left|\frac{\lambda_{v_1}\lambda_{v_2}}{\lambda_{v_3}\lambda_{v_4}}\right|} f(\beta). \tag{10.4}$$

Using Lemma 7.1, we have

$$|\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}\rangle| = \sum_\beta \sqrt{\left|\frac{\lambda_{k_1}\lambda_{k_2}}{\lambda_{k_3}\lambda_{k_4}}\right|} \prod_{i=1}^2 \sqrt{(\beta(k_i)+1)} \prod_{i=3}^4 \sqrt{\beta(k_i)} |f(\beta)|^2. \tag{10.5}$$

Consider first the case $k_1 = k_2$ and, by (10.1), $k_3 \neq k_4$. Using the estimates (5.11) for λ_{k_i}, we have

$$|\langle a_{k_1}^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4}\rangle| = |\lambda_{v_3}\lambda_{v_4}|^{-\frac{1}{2}} \rho^{-\frac{1}{10}} (Q_\Psi(k_1,k_3,k_4) + Q_\Psi(k_3,k_4)). \tag{10.6}$$

Since $\sum_{k_1} Q_\Psi(k_1,k_3,k_4) \le N Q_\Psi(k_3,k_4)$, we have

$$\sum_{k_1} |\langle a_{k_1}^\dagger a_{k_1}^\dagger a_{k_3} a_{k_4}\rangle| = |\lambda_{v_3}\lambda_{v_4}|^{-\frac{1}{2}} \rho^{-\frac{1}{10}} (N Q_\Psi(k_3,k_4) + \Lambda k_c^3 Q_\Psi(k_3,k_4)).$$

With $k_3 \neq \pm k_4$ and the bound on $Q_\Psi(k_3,k_4)$ in (5.42), we arrive at the desired result (10.2).

The case $k_1 = -k_3$ can be proved in a similarly way by using

$$\sqrt{(\beta(k_1)+1)(\beta(k_2)+1)} \le \tfrac{1}{2}(\beta(k_2)+\beta(k_1)+2). \qquad \square$$

By symmetry, we can prove some other special cases such as $k_1 = -k_4$ are negligible. So from now on we focus on the cases

$$k_1 + k_2 = k_3 + k_4, \quad k_i \in P_{H,c}, \quad k_i \neq \pm k_j \quad \text{for } i \neq j. \tag{10.7}$$

This condition will be imposed for the rest of this section. Denote by $M[k_1,k_2]$ the set of all states created by a soft pair creation $A^{k_1+k_2,\ k_1/2-k_2/2}$ from another state, i.e.,

$$M(k_1,k_2) \equiv \{\beta \in M | \exists \alpha \in M^s_{k_1+k_2} \text{ such that } \mathcal{A}^{k_1+k_2,\ k_1/2-k_2/2}\alpha = \beta\} \tag{10.8}$$

if $k_1 + k_2 \in P_L$. Otherwise, we set $M[k_1,k_2] = \emptyset$. Notice that

$$|\mathcal{A}^{k_1+k_2,\ k_1/2-k_2/2}\alpha\rangle = C a_{k_1}^\dagger a_{k_2}^\dagger a_{k_1+k_2} a_0 |\alpha\rangle$$

for some normalization constant C. Hence for $\beta, \gamma \in M$, if

$$\langle\beta|a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}|\gamma\rangle \neq 0,$$

we have $k_1+k_2=k_3+k_4$ and

$$\mathcal{A}^{k_1+k_2,\ k_1/2-k_2/2}\alpha=\beta \quad \Leftrightarrow \quad \mathcal{A}^{k_3+k_4,\ k_3/2-k_4/2}\alpha=\gamma. \tag{10.9}$$

The main contribution of the four nonvanishing leg term is identified in the next lemma.

Lemma 10.2

$$\lim_{k_c,\rho}\rho^{-5/2}|\Lambda|^{-2}\sum_{(10.7)}\sum_{\beta\in M(k_1,k_2)}V_{k_1-k_3}f(\beta)f(\gamma)\langle\beta|a^\dagger_{k_1}a^\dagger_{k_2}a_{k_3}a_{k_4}|\gamma\rangle$$
$$\le 4\|w^2V\|_1\frac{1}{3\pi^2}g_0^{3/2}. \tag{10.10}$$

Proof By (10.9), we have

$$\sum_{\beta\in M(k_1,k_2)} f(\beta)f(\gamma)\langle\beta|a^\dagger_{k_1}a^\dagger_{k_2}a_{k_3}a_{k_4}|\gamma\rangle$$
$$=\prod_{i=1}^4\sqrt{\lambda_{k_i}}\sum_{\alpha\in M^s_{k_1+k_2}}4|f(\alpha)|^2|\Lambda|^{-2}\alpha(0)\alpha(k_1+k_2)\prod_{i=1}^4\sqrt{(\alpha(k_i)+1)}. \tag{10.11}$$

We claim that (10.11) is very close to the following expression:

$$\prod_{i=1}^4\sqrt{\lambda_{k_i}}\sum_{\alpha\in M^s_{k_1+k_2}}4|f(\alpha)|^2|\Lambda|^{-2}\alpha(0)\alpha(k_1+k_2). \tag{10.12}$$

For $x_i\ge 0$, we have

$$1\le\sqrt{(x_1+1)(x_2+1)(x_3+1)(x_4+1)}\le\frac{1}{4}(x_1+x_2+2)(x_3+x_4+2). \tag{10.13}$$

Since $\alpha(0)\le N$ and $\alpha(k_1+k_2)\le m_c$, we have

$$\frac{|(10.11)-(10.12)|}{\left|\prod_{i=1}^4\sqrt{\lambda_{k_i}}\right|}\le\frac{4m_c\rho}{|\Lambda|}\left(\sum_i Q_\Psi(k_i)+\sum_{i,j}Q_\Psi(k_i,k_j)\right)\le\frac{\rho^2}{|\Lambda|} \tag{10.14}$$

where we have used (5.15) and (5.42).

By definition, $Q_\Psi(0,k_1+k_2)=\sum_{\alpha\in M}\alpha(0)\alpha(k_1+k_2)$. Together with $\alpha(0)\le N$ and $\alpha(k_1+k_2)\le m_c$, we have

$$\left|\frac{(10.12)}{\prod_{i=1}^4\sqrt{\lambda_{k_i}}}-4|\Lambda|^{-2}Q_\Psi(0,k_1+k_2)\right|\le\frac{4m_c\rho}{|\Lambda|}\sum_{\alpha\in M^a_{k_1+k_2}}|f(\alpha)|^2. \tag{10.15}$$

Using the bound (5.23) concerning $\sum_{\alpha\in M^a_{k_1+k_2}}$, we have

$$\left|\frac{(10.12)}{\prod_{i=1}^4\sqrt{\lambda_{k_i}}}-4|\Lambda|^{-2}Q_\Psi(0,k_1+k_2)\right|\le\rho^{3/2}|\Lambda|^{-1}. \tag{10.16}$$

Combining (10.14), (10.16), with the bounds on λ in (5.11), we have:

$$\left|(10.11) - \prod_{i=1}^{4} \sqrt{\lambda_{k_i}} 4|\Lambda|^{-2} Q_\Psi(0, k_1 + k_2)\right| \le \frac{\rho^{5/4}}{|\Lambda|}. \qquad (10.17)$$

Since $\lambda_p = -w_p = -g_p p^{-2}$ for $p \in P_H$ and $|g_p - g_q| \le \text{const.}\, ||p| - |q||$, we have for $p, q \in P_{H,c}$ with $p + q \in P_L$

$$|\lambda_p - \lambda_q| \le \text{const.}\, \varepsilon_H^{-1} ||p| - |q|| \le \rho^{3/4}. \qquad (10.18)$$

This implies $||\sqrt{\lambda_p}| - |\sqrt{\lambda_q}|| \le \rho^{3/8}$. Applying these results to $\prod_{i=1}^4 \sqrt{\lambda_{k_i}}$ with $k_1 + k_2 = k_3 + k_4 \in P_L$, we have

$$\left|\prod_{i=1}^{4} \sqrt{\lambda_{k_i}} - \lambda_{k_1}\lambda_{k_3}\right| \le \rho^{1/4}. \qquad (10.19)$$

Inserting this inequality into (10.17) and using $Q_\Psi(0, k_1 + k_2) \le N m_c$, we obtain

$$|(10.11) - 4\lambda_{k_1}\lambda_{k_3}|\Lambda|^{-2} Q_\Psi(0, v)| \le \rho^{5/4} m_c |\Lambda|^{-1}, \quad v = k_1 + k_2. \qquad (10.20)$$

Summing over $v \in P_L$ and $k_1, k_3 \in P_{H,c}$, we have that the left hand side of (10.10) is equal to

$$\lim_{k_c \to \infty, \rho \to 0} 4\|w^2 V\|_1 \sum_{v \in P_L} Q_\Psi(0, v) \rho^{-5/2} |\Lambda|^{-2}. \qquad (10.21)$$

With (8.28), we have proved (10.10). □

11 Interaction Energy with Four High Momentum Legs II: The Error Terms

Our goal in this section is to prove that the interaction energy associated with four high momentum legs which are not covered by Lemma 10.2 is negligible. We state it as the following lemma. Notice that Lemma 4.6 follows from the results in the previous two sections and this lemma.

Lemma 11.1

$$\lim_{k_c, \rho} \sum_{(10.7)} \sum_{\beta \notin M(k_1,k_2)} \left|\frac{V_{k_1 - k_3}}{|\Lambda|} f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle\right| (\rho^{5/2}\Lambda)^{-1} = 0. \qquad (11.1)$$

We start with the following lemma.

Lemma 11.2

$$\lim_{k_c, \rho} \sum_{(10.7)} \sum_{\beta, \gamma : \beta \notin M(k_1,k_2)} |f(\beta) f(\gamma)| \le \Lambda \le o(\rho^{5/2}|\Lambda|^2) \qquad (11.2)$$

where the summation is restricted to all $\beta, \gamma \in M$ such that

$$\langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle \ne 0. \qquad (11.3)$$

Proof In this section, we use the following notations:

$$\mathcal{A}_{-k,k}\alpha \equiv \mathcal{A}^k\alpha \quad \text{and} \quad \mathcal{A}_{-k+\frac{u}{2},k+\frac{u}{2}}\alpha \equiv \mathcal{A}^{u,k}\alpha. \tag{11.4}$$

For any $\{v_1,\dots,v_t\}\subset P_L$ such that $v_i\neq \pm v_j$, $1\le i,j\le t$ and $\alpha\in M^s_{v_i}$, $1\le i\le t$, define

$$M(\alpha,s,\{v_1,\dots,v_t\})\equiv\left\{\prod_{i=1}^{t+s}\mathcal{A}_{q_i,q_i'}\alpha,\, q_i,q_i'\in P_{H,c},\, q_i+q_i'=u_i\right\} \tag{11.5}$$

where $u_i=v_i$, $1\le i\le t$ and $u_i=0$ otherwise. Since $v_i\in P_L$ and all other momenta are in $P_{H,c}$, $\mathcal{A}_{q_i,q_i'}$'s commutes with one another.

Proposition 11.1 *For any $\chi\in M$, there exists $(\alpha,s,\{v_1,\dots,v_t\})$ such that*

$$\chi\in M(\alpha,s,\{v_1,\dots,v_t\}). \tag{11.6}$$

Proof By definition of M, we can write the state $|\chi\rangle$ as follows:

$$|\chi\rangle=\prod_{i=1}^{t}\mathcal{A}_{p_i,p_i'}\prod_{k=1}^{s}\mathcal{A}_{q_k,-q_k}\prod_{j=1}^{w}(\mathcal{A}_{u_j,-u_j})^{n_j}|N\rangle, \tag{11.7}$$

where $u_j\notin P_{H,c}$, $v_i:=p_i+p_i'\in P_L$, $p_i,p_i',q_k\in P_{H,c}$. Furthermore, we require that $u_j\neq \pm u_{j'}$ for $j\neq j'$ and $v_i\neq\pm v_{i'}$ for $i\neq i'$. Notice that $\mathcal{A}_{p,p'}$ commute with $\mathcal{A}_{q,-q}$ so that their orderings are not important. Clearly, the choice of

$$\alpha=\prod_{j=1}^{w}(\mathcal{A}_{u_j,-u_j})^{n_j}|N\rangle \tag{11.8}$$

yields that $\chi\in M(\alpha,s,\{v_1,v_2,\dots,v_t\})$ and this proves the proposition. □

For any β,γ satisfying (11.3), we have $\beta(u)=\gamma(u)$ for $u\in P_L\cup P_I\cup P_0$. From the proof of Proposition 11.1, there exists $(\alpha,s,\{v_i,1\le i\le t\})$ such that

$$\beta \text{ and } \gamma\in M(\alpha,s,\{v_1,\dots,v_t\}). \tag{11.9}$$

Notice α is the same for both β and γ and $\alpha\in M^s_u$ for any $u\in P_L$.

For any $(\alpha,s,\{v_1,\dots,v_t\})$, define $N(\alpha,s,\{v_1,\dots,v_t\})$ as the set of the pairs (β,γ) such that

1. $\beta,\gamma\in M(\alpha,s,\{v_1,\dots,v_t\})$
2. there exist k_i, $i=1,\dots,4$ satisfying (10.7), $\beta\notin M(k_1,k_2)$ and (11.3) holds
3. for any other $\alpha',s',\{v_1',\dots,v_{t'}'\}$ s.t. $\beta,\gamma\in M(\alpha',s',\{v_1',\dots,v_{t'}'\})$, then

$$s+t\le s'+t'. \tag{11.10}$$

We assume $(\beta,\gamma)\in M(\alpha,s,\{v_1,\dots,v_t\})$ and (11.3) holds. Clearly, $s+t=1$ or $t=0$ implies that $\beta\in M[k_1,k_2]$. Hence if $N(\alpha,s,\{v_1,\dots,v_t\})$ is not an empty set then

$$s+t\ge 2 \quad \text{and} \quad t\ge 1. \tag{11.11}$$

By definition of $N(\alpha, s, \{v_1, \dots, v_t\})$, we have

$$\sum_{(10.7)} \sum_{\beta \notin M(k_1,k_2)} |f(\beta) f(\gamma)| \le \sum_{\alpha, s, \{v_1,\dots,v_t\}} |N(\alpha, s, \{v_1, \dots, v_t\})| \max_{\beta,\gamma \in M(\alpha,s,\{v_1,\dots,v_t\})} |f(\beta) f(\gamma)|, \tag{11.12}$$

where $|N(\alpha, s, \{v_1, \dots, v_t\})|$ is the cardinality of $N(\alpha, s, \{v_1, \dots, v_t\})$. By definition of f, if $\beta, \gamma \in M(\alpha, s, \{v_1, \dots, v_t\})$ then

$$|f(\beta) f(\gamma)| \le \left|\frac{\alpha(0)}{|\Lambda|}\right|^{2s+t} \left|\frac{m_c}{|\Lambda|}\right|^{t} \max_{k \in P_H} \{\lambda_k\}^{2t+s} |f(\alpha)|^2. \tag{11.13}$$

From (5.11) and $m_c = \rho^{-\eta}$, we have

$$\max_{\beta,\gamma \in M(\alpha,s,\{v_1,\dots,v_t\})} |f(\beta) f(\gamma)| \le (\text{const.}\, \rho^{1-5\eta})^{2s+t} |\Lambda|^{-t} |f(\alpha)|^2.$$

Together with (11.12), the right hand side of (11.12) is bounded by

$$\le \sum_{\alpha, s, \{v_1,\dots,v_t\}} |N(\alpha, s, \{v_1, \dots, v_t\})| (\text{const.}\, \rho^{1-5\eta})^{2s+t} |\Lambda|^{-t} |f(\alpha)|^2. \tag{11.14}$$

Define $N(\alpha, s, t)$ and $N(s, t)$ by

$$N(\alpha, s, t) \equiv \max_{\{v_1,\dots,v_t\}} \{|N(\alpha, s, \{v_1, \dots, v_t\})|\}, \tag{11.15}$$

$$N(s, t) \equiv \max_{\alpha} \{N(\alpha, s, t)\}. \tag{11.16}$$

With (11.14), we can bound (11.12) by

$$\begin{aligned}(11.12) &\le \sum_{\alpha,s,t} |f(\alpha)|^2 \sum_{\{v_1,\dots,v_t\}} N(\alpha, s, t) (\text{const.}\, \rho^{1-5\eta})^{2s+t} |\Lambda|^{-t} \\ &\le \sum_{s,t} \sum_{\{v_1,\dots,v_t\}} N(s, t) (\text{const.}\, \rho^{1-5\eta})^{2s+t} |\Lambda|^{-t}. \end{aligned} \tag{11.17}$$

For fixed t the total number of set $\{v_1, \dots, v_t, v_i \in P_L\}$ is bounded by

$$\sum_{\{v_1,\dots,v_t\}} 1 \le (\Lambda \rho^{3/2} \eta_L^{-3})^t (t!)^{-1} \le (\rho^{1-5\eta})^{\frac{3t}{2}} |\Lambda|^t (t!)^{-1}$$

From $t \le (\Lambda \rho^{3/2} \eta_L^{-3}) \le \rho^{-1.65}$ and (11.11), we have

$$\sum_{(10.7)} \sum_{\beta \notin M(k_1,k_2)} |f(\beta) f(\gamma)| \le \sum_{t=1}^{\rho^{-1.65}} \sum_{s: s+t \ge 2} N(s, t) (\text{const.}\, \rho^{1-5\eta})^{2s+\frac{5t}{2}} (t!)^{-1}. \tag{11.18}$$

Lemma 11.3 *For any $N(\alpha, s, \{v_1, \dots, v_t\})$, $s + t \ge 2$ and $t \ge 1$, we have*

$$|N(\alpha, s, \{v_1, \dots, v_t\})| \le t!\, t^{(\frac{t}{2})} |\Lambda|^{\frac{s+t}{2}+1} (\rho^{-5\eta})^{t+s}. \tag{11.19}$$

From this lemma and $\Lambda = \rho^{-\frac{25}{8}}$, the right hand side of (11.18) is bounded above by

$$\sum_{t\geq 1}^{\rho^{-1.65}} \sum_{s:s+t>1} (\rho^{5/2}|\Lambda|^{1/2}t^{1/2})^t (\rho^2|\Lambda|^{1/2})^s (\text{const. } \rho^{-35\eta/2})^{t+s} \Lambda$$
$$= \sum_{t\geq 1}^{\rho^{-1.65}} (\text{const. } \rho^{0.85} t^{1/2})^t \sum_{s:s+t>1} (\text{const. } \rho^{0.35})^s \Lambda \leq \Lambda.$$

This proves Lemma 11.2. □

We now prove Lemma 11.3.

Proof Since $(\beta, \gamma) \in N(\alpha, s, \{v_1, \ldots, v_t\})$, we can express them as

$$\beta = \prod_{j=t+1}^{s+t} \mathcal{A}_{q_{2j-1},q_{2j}} \prod_{i=1}^{t} A_{q_{2i-1},q_{2i}} \alpha, \qquad \gamma = \prod_{j=t+1}^{s+t} \mathcal{A}_{\widetilde{q}_{2j-1},\widetilde{q}_{2j}} \prod_{i=1}^{t} A_{\widetilde{q}_{2i-1},\widetilde{q}_{2i}} \alpha \tag{11.20}$$

and $q_{2i-1} + q_{2i} = v_i = \widetilde{q}_{2i-1} + \widetilde{q}_{2i}$ for $i = 1, \ldots, t$, $q_{2j-1} + q_{2j} = \widetilde{q}_{2j-1} + \widetilde{q}_{2j} = 0$ for $t+1 \leq j \leq s+t$. From (11.3), we have

$$\{q_1, \ldots, q_{2s+2t}\} - \{k_1, k_2\} = \{\widetilde{q}_1, \ldots, \widetilde{q}_{2s+2t}\} - \{k_3, k_4\}. \tag{11.21}$$

Denote the common elements in $\{q_i\}$ and $\{\widetilde{q}_i\}$ by $p_1, p_2, \ldots, p_{2s+2t-2}$. Then we have

$$\{q_i\} = k_1, \ k_2, \ p_1, \ p_2, \ \ldots, \ p_{2s+2t-2}, \tag{11.22}$$
$$\{\widetilde{q}_i\} = k_3, \ k_4, \ p_1, \ p_2, \ \ldots, \ p_{2s+2t-2}. \tag{11.23}$$

We now construct a graph with vertices $\{k_1, k_2, k_3, k_4, p_i, 1 \leq i \leq 2s+2t-2\}$. The edges of the graphs consisting of β edges (q_{2i-1}, q_{2i}), $1 \leq i \leq s+t$ and γ edges $(\widetilde{q}_{2j-1}, \widetilde{q}_{2j})$, $1 \leq i \leq s+t$. From (11.3), the graph can be decomposed into two chains and loops. Thus there exist $l, m_i \in \mathbb{Z}$ and $0 < m_1 < m_2 < \cdots < m_l = s+t$ such that

$$\begin{aligned}
&k_1 \longleftrightarrow p_1 \longleftrightarrow p_2 \longleftrightarrow p_3 \cdots p_{2m_1-1} \longleftrightarrow k_2 \ (\text{or } \ \cdots k_4) \\
&k_3 \longleftrightarrow p_{2m_1} \longleftrightarrow p_{2m_1+1} \cdots p_{2m_2-2} \longleftrightarrow k_4 \ (\text{or } \ \cdots k_2) \\
&p_{2m_2-1} \longleftrightarrow p_{2m_2} \longleftrightarrow p_{2m_2+1} \cdots p_{2(m_3)-2} \longleftrightarrow p_{2m_2-1} \\
&\ldots \\
&\ldots \\
&p_{2m_{l-1}-1} \longleftrightarrow p_{2m_{l-1}} \longleftrightarrow p_{2m_{l-1}+1} \cdots p_{2(m_l)-2} \longleftrightarrow p_{2m_{l-1}-1}.
\end{aligned} \tag{11.24}$$

Here we have relabeled the indices of p and do not distinguish β edges and α edges. We also disregard the obvious symmetry $k_1 \to k_2$ and $k_3 \to k_4$. Due to the condition (11.10), the length of the loop must be 4 or more, i.e., for $3 \leq i \leq l$

$$m_{i-1} + 2 \leq m_i. \tag{11.25}$$

Together with $m_l = s + t$, we obtain

$$l \le (s+t)/2 + 1, \quad t \ge 1. \tag{11.26}$$

Without loss of generality, we assume for $3 \le i < j \le l$

$$m_i - m_{i-1} \le m_j - m_{j-1}. \tag{11.27}$$

Denote by $N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})$ the set of all pairs (β, γ) having the graph above and we now estimate its cardinality.

We can add the information between k_i's and p_i's as follows

$$\begin{aligned}
&k_1 \overset{w_1}{\longleftrightarrow} p_1 \overset{\widetilde{w}_1}{\longleftrightarrow} p_2 \overset{w_2}{\longleftrightarrow} p_3 \cdots p_{2m_1-1} \overset{w_{m_1}}{\longleftrightarrow} k_4 \text{ (or } \cdots k_2) \\
&k_3 \overset{\widetilde{w}_{m_1}}{\longleftrightarrow} p_{2m_1} \overset{w_{m_1+1}}{\longleftrightarrow} p_{2m_1+1} \cdots p_{2m_2-2} \overset{\widetilde{w}_{m_2}}{\longleftrightarrow} k_2 \text{ (or } \cdots k_4) \\
&p_{2m_2-1} \overset{w_{m_2+1}}{\longleftrightarrow} p_{2m_2} \overset{\widetilde{w}_{m_2+1}}{\longleftrightarrow} p_{2m_2+1} \cdots p_{2(m_3)-2} \overset{\widetilde{w}_{m_3}}{\longleftrightarrow} p_{2m_2-1} \\
&\dots \\
&\dots \\
&p_{2m_{l-1}-1} \overset{w_{m_{l-1}+1}}{\longleftrightarrow} p_{2m_{l-1}} \overset{\widetilde{w}_{m_{l-1}+1}}{\longleftrightarrow} p_{2m_{l-1}+1} \cdots p_{2(m_l)-2} \overset{\widetilde{w}_{m_l}}{\longleftrightarrow} p_{2m_{l-1}-1},
\end{aligned} \tag{11.28}$$

where $A \overset{c}{\longleftrightarrow} B$ if and only if $A + B = c$. And w_i's the union of s zero's and $\{v_1, \dots, v_t\}$, so are $\widetilde{w}$'s. By (11.20), β and γ is uniquely determined by w_i's, $\widetilde{w}_i$'s and one k_i or p_i for each loop or chain.

To bound $|N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})|$, we note that the sum of momentum in each loop is zero. Thus we can count the number of graphs as follows.

1. choose the positions of zeros in β edges. The total number of choices is less than 2^{t+s}
2. choose the positions of $v_1 \cdots v_t$ in β edges. The total number of choices is $t!$
3. choose the positions of zeros in γ edges. The total number of choices is less than 2^{t+s}
4. choose the positions of $v_1 \cdots v_t$ in γ edges. We call a loop trivial if all the momenta associated with γ edges are zero. The number of trivial loops is at most $s/2$ since there are at least two γ edges per loop. Hence the number of non-trivial loops is at least $l - s/2$. Thus we only have to fix v in at most $t - (l - s/2)$ edges and the number of choices is at most $t^{t-l+s/2}$

Thus we obtain

$$\begin{aligned}
&|N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})| \\
&\quad \le (\text{const.})^{t+s} t! t^{(t+s/2-l)} \left(k_c^3 \Lambda\right)^l \\
&\quad \le (\text{const.})^{t+s} t! t^{(t/2)} \left(k_c^3 \Lambda\right)^{t/2+s/2+1}
\end{aligned} \tag{11.29}$$

where we have used (11.26) Since

$$|N(\alpha, s, \{v_1, \dots, v_t\})| = \sum_l \sum_{\{m_1, \dots, m_l\}} |N(\alpha, s, \{v_1, \dots, v_t\}, l, \{m_1, \dots, m_l\})|$$

and

$$\sum_{l} \sum_{\{m_1,\dots,m_l\}} 1 \le \text{const.}^{s+t} \tag{11.30}$$

we have proved (11.19). □

We now prove Lemma 11.1.

Proof Let $\beta, \gamma \in M$ s.t. $\langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle \neq 0$. Using Lemma 7.1 and the definition of f, we have

$$\begin{aligned} |f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle| &= f(\beta) f(\gamma) \sqrt{\beta(k_1)\beta(k_2)\gamma(k_3)\gamma(k_4)} \\ &\le |f(\beta)|^2 \sqrt{\left| \frac{\lambda_{k_3}\lambda_{k_4}}{\lambda_{k_1}\lambda_{k_2}} \right|} (\beta(k_1) + \beta(k_2)) \sqrt{\gamma(k_3)\gamma(k_4)}. \end{aligned} \tag{11.31}$$

From the bound on λ_{k_i}'s in (5.11) and $N = \rho^{-17/8}$, we have

$$|f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle| \le |f(\beta)|^2 (\lambda_{k_1}\lambda_{k_2})^{-\frac{1}{2}} (\beta(k_1) + \beta(k_2)) \rho^{-\frac{9}{4}}.$$

Since $Q_\Psi(\{k, m\})$ decays exponentially with m for $k \in P_H$ (5.40), we have

$$\sum_{(10.7)} \sum_{\beta(k_1)>3 \text{ or } \beta(k_2)>3} |f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle| \le o(\rho^{5/2} |\Lambda|^2). \tag{11.32}$$

By symmetry, we have

$$\sum_{(10.7)} \sum_{\gamma(k_3)>3 \text{ or } \gamma(k_4)>3} |f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle| \le o(\rho^{5/2} |\Lambda|^2). \tag{11.33}$$

To prove (11.1), we only have to focus on the case $\beta(k_i) \le 3, i = 1, 2$ and $\gamma(k_i) \le 3, i = 3, 4$. In this case, by (11.31), we have

$$|f(\beta) f(\gamma) \langle \beta | a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} | \gamma \rangle| \le |\text{const.}\, f(\beta) f(\gamma)|. \tag{11.34}$$

Using Lemma 11.2, we arrive at the desired result (11.1). □

12 Proof of Lemma 2.2

The proof of Lemma 2.2 is standard and only a sketch will be given. We first construct an isometry between functions with periodic boundary condition in $[0, L]^3$ and functions with Dirichlet boundary condition in $[-\ell, L+\ell]^3$. Denote the coordinates of x by $\mathrm{x} = (x^{(1)}, x^{(2)}, x^{(3)})$. Let $h(\mathrm{x})$ supported on $[-\ell, L+\ell]^3$ be the function $h(\mathrm{x}) = q(x^{(1)}) q(x^{(2)}) q(x^{(3)})$ where

$$q(x) = \begin{cases} \cos[(x-\ell)\pi/4\ell], & |x| \le \ell, \\ 1, & \ell < x < L - \ell, \\ \cos[(x-(L-\ell))\pi/4\ell], & |x - L| \le \ell, \\ 0, & \text{otherwise}. \end{cases} \tag{12.1}$$

The function $q(x)$ is symmetric w.r.t. $x = L/2$. Due to the property of cosine, for any function ϕ with the period L we have

$$\int_{\mathrm{x}\in[-\ell,\,L+\ell]^3} |h\phi(\mathrm{x})|^2 = \int_{\mathrm{x}\in[0,L]^3} |\phi(x)|^2. \tag{12.2}$$

Thus the map $\phi \longrightarrow h\phi$ is an isometry:

$$L^2_{\text{Periodic}}([0,L]^3) \to L^2_{\text{Dirichlet}}([-\ell, L+\ell]^3).$$

Let $\chi(\mathrm{x})$ be the characteristic function of the ℓ-boundary of $[0,L]^3$, i.e., $\chi(\mathrm{x}) = 1$ if $|x^{(\alpha)}| \le \ell$ for some $\alpha = 1, 2$ or 3 where $|x^{(\alpha)}|$ is the distance on the torus. Then standard methods yield the following estimate on the kinetic energy of $h\phi$

$$\begin{aligned}&\int_{\mathrm{x}\in[-\ell,\,L+\ell]^3} |\nabla(h\phi)(\mathrm{x})|^2 \\ &\quad \le \int_{\mathrm{x}\in[0,\,L]^3} |\nabla\phi(\mathrm{x})|^2 + \text{const.}\,\ell^{-2}\int \chi(\mathrm{x})|\phi(\mathrm{x})|^2.\end{aligned} \tag{12.3}$$

The generalization of this isometry to higher dimensions is straightforward. Suppose $\Psi(\mathrm{x}_1, \ldots, \mathrm{x}_N)$ is a function with period L. Then for any $u \in \mathbb{R}^3$, the map

$$\mathcal{F}^u(\Psi) := \Psi(\mathrm{x}_1, \ldots, \mathrm{x}_N)\prod_{i=1}^N h(\mathrm{x}_i + u) \tag{12.4}$$

is an isometry from $L^2_{\text{Periodic}}([0,L]^{3N})$ to $L^2_{\text{Dirichlet}}([-\ell - u, L + \ell - u]^{3N})$. Clearly, $\mathcal{F}^u$ has the property (12.3).

The potential V can be extended to be periodic by defining $V^P(x-y) = V([x-y]_P)$ where $[x-y]_P$ is the difference of x and y as elements on the torus $[0,L]$. Since V is nonnegative and has fast decay in the position space, we have $V(x-y) \le V^P(x-y)$. From the definition of $\mathcal{F}^u$, we conclude that

$$\int |\mathcal{F}^u(\Psi)|^2 V(\mathrm{x}_1 - \mathrm{x}_2)\prod_{i=1}^N d\mathrm{x}_i \le \int_{[0,\,L]^{3N}} |\Psi|^2 V^P(\mathrm{x}_1 - \mathrm{x}_2)\prod_{i=1}^N d\mathrm{x}_i.$$

Therefore, the energy of two boundary conditions are related by

$$\langle H_N\rangle_{F^u(\Psi)} \le \langle H_N\rangle_\Psi + \text{const.}\,\ell^{-2}\sum_{i=1}^N \langle \chi(\mathrm{x}_i + u)\rangle_\Psi\,. \tag{12.5}$$

Averaging over $u \in [0,L]^3$, we have

$$\int_{[0,L]^3} \langle H_N\rangle_{F^u(\Psi)}\, du \le L^3\,\langle H_N\rangle_\Psi + \text{const.}\,\ell^{-1}L^2 N. \tag{12.6}$$

So for any Ψ there exists an u such that

$$\langle H_N\rangle_{F^u(\Psi)} \le \langle H_N\rangle_\Psi + \text{const.}\,N\left(\frac{1}{\ell L}\right). \tag{12.7}$$

If we choose ℓ and L as

$$\ell = \rho^{-25/48}, \qquad L = \rho^{-25/24}, \tag{12.8}$$

the error term is negligible to the accuracy we need in proving Lemma 2.2. This concludes the proof of Lemma 2.2. □

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